Efficient Rational Creative Telescoping

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Joint work with Mark Giesbrecht, George Labahn and Eugene Zima

Outline

- Creative telescoping
- New approach for rational functions

Hypergeometric summations

Hypergeometric summations

Consider

$$\sum_{k=0}^{n} f(n,k),$$

where $f(n,k)$ typically involves

$$\frac{1}{(k+1)(k+2)}, \quad 2^k, \quad k!, \quad \binom{n}{k}, \quad \dots.$$

Hypergeometric summations/identities

Consider

$$\sum_{k=0}^{n} f(n,k) = F(n),$$

where $f(n,k)$ typically involves

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Hypergeometric summations/identities

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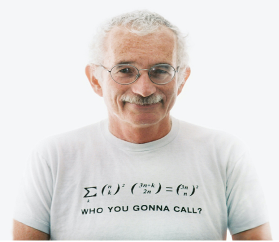
$$\frac{1}{(k+1)(k+2)}, \quad 2^k, \quad k!, \quad \binom{n}{k}, \quad \ldots.$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$$



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where $f(n,k)$ typically involves

$$\frac{1}{(k+1)(k+2)}, \quad 2^k, \quad k!, \quad \binom{n}{k}, \quad \ldots.$$

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$$

Hypergeometric summations/identities

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$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k} = \frac{(n+b+c)!}{n!b!c!}$$

Hypergeometric summations/identities

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where $f(n,k)$ typically involves

$$\frac{1}{(k+1)(k+2)}, \quad 2^k, \quad k!, \quad \binom{n}{k}, \quad \ldots.$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} = \begin{bmatrix} b+n \\ n \end{bmatrix}_q$$

Hypergeometric summations/identities

Consider

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where $f(n,k)$ typically involves

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$$\sum_{k=-\infty}^{\infty} (-1)^k \begin{bmatrix} n+b \\ n+k \end{bmatrix}_q \begin{bmatrix} n+c \\ c+k \end{bmatrix}_q \begin{bmatrix} b+c \\ b+k \end{bmatrix}_q q^{k(3k-1)/2} = \begin{bmatrix} n+b+c \\ n, b, c \end{bmatrix}_q$$

The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ s.t.

$$f(n,k) \;=\; g(n,k+1) - g(n,k).$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$F(n) \;= \sum_{k=0}^{n} \big(g(n,k+1) - g(n,k)\big).$$

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The creative telescoping problem

GIVEN $k \cdot k!$, FIND $k!$ s.t.

$$k \cdot k! \; = \; (k+1)! - k!.$$

Then $F(n) = \sum_{k=0}^{n} k \cdot k!$ satisfies

$$F(n) \; = (n+1)! - 1.$$

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The creative telescoping problem

GIVEN $f(n,k)$, FIND $c_0(n),\ldots,c_\rho(n)$ and $g(n,k)$ s.t.

$$c_0(n)f(n,k)+\cdots+c_\rho(n)f(n+\rho,k) \;=\; g(n,k+1)-g(n,k).$$

Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n).$$

The creative telescoping problem

GIVEN $\binom{n}{k}$, FIND $-2, 1$ and $-\binom{n}{k-1}$ s.t.

$$-2\binom{n}{k} + \binom{n+1}{k} = -\binom{n}{k} - \left(-\binom{n}{k-1}\right).$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n+1) = 0.$$

The creative telescoping problem

GIVEN $f(n, k)$, FIND $c_0(n), \ldots, c_\rho(n)$ and $g(n, k)$ s.t.

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Notation. $S_n(f(n,k))=f(n+1,k)$ and $S_k(f(n,k))=f(n,k+1)$.

The creative telescoping problem

GIVEN $f(n,k)$, FIND $c_0(n), \ldots, c_\rho(n)$ and $g(n,k)$ s.t.

$$\big(c_0(n) + \cdots + c_\rho(n) S_n^\rho\big)\big(f(n,k)\big) \;=\; (S_k - 1)\big(g(n,k)\big)$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n).$$

Notation. $S_n(f(n,k)) = f(n+1,k)$ and $S_k(f(n,k)) = f(n,k+1)$.

The creative telescoping problem

GIVEN $f(n,k)$, FIND $c_0(n), \ldots, c_\rho(n)$ and $g(n,k)$ s.t.

$$\underbrace{\left(c_0(n) + \cdots + c_\rho(n) S_n^\rho\right)}_{\text{telescoper}} \big(f(n,k)\big) = (S_k - 1)\big(\underbrace{g(n,k)}_{\text{certificate}}\big)$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n).$$

Notation. $S_n(f(n,k)) = f(n+1,k)$ and $S_k(f(n,k)) = f(n,k+1)$.

Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
4. The reduction-based approach (since $\approx$ 2010)

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A reduction-based approach (CHKL2015)

Example. $\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}$

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$$f = (S_k - 1)\big(g_0\big) + \frac{nk}{(n+2k)^2+2}$$

$\underbrace{\qquad\qquad}_{\text{summable}}$ $\underbrace{\qquad\qquad}_{\text{remainder}}$

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$$f = (S_k - 1)(g_0) + \frac{nk}{(n+2k)^2+2}$$

$$\frac{1}{nk+1} + \sum_{j=1}^{10} \frac{n(k+j)}{(n+2k+2j)^2+2}$$

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$$S_n(f) = (S_k - 1)\big(g_1\big) + \frac{(n+1)k}{(n+2k+1)^2+2}$$

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$$c_0(n)\, f = (S_k - 1)\big(c_0(n)\, g_0\big) + c_0(n)\, \frac{nk}{(n+2k)^2+2}$$

$$c_1(n)\, S_n(f) = (S_k - 1)\big(c_1(n)\, g_1\big) + c_1(n)\, \frac{(n+1)k}{(n+2k+1)^2+2}$$

$$c_2(n)\, S_n^2(f) = (S_k - 1)\big(c_2(n)\, g_2\big) + c_2(n)\, \frac{(n+2)(k-1)}{(n+2k)^2+2}$$

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$$+\left\{\begin{aligned} c_0(n)\, f &= (S_k - 1)\big(c_0(n)\, g_0\big) + c_0(n)\, \tfrac{nk}{(n+2k)^2+2} \\ c_1(n)\, S_n(f) &= (S_k - 1)\big(c_1(n)\, g_1\big) + c_1(n)\, \tfrac{(n+1)k}{(n+2k+1)^2+2} \\ c_2(n)\, S_n^2(f) &= (S_k - 1)\big(c_2(n)\, g_2\big) + c_2(n)\, \tfrac{(n+2)(k-1)}{(n+2k)^2+2} \\ c_3(n)\, S_n^3(f) &= (S_k - 1)\big(c_3(n)\, g_3\big) + c_3(n)\, \tfrac{(n+3)(k-1)}{(n+2k+1)^2+2} \\ c_4(n)\, S_n^4(f) &= (S_k - 1)\big(c_4(n)\, g_4\big) + c_4(n)\, \tfrac{(n+4)(k-2)}{(n+2k)^2+2} \end{aligned}\right.$$

$$c_0(n)\, f + \cdots + c_4(n)\, S_n^4(f) = (S_k - 1)\Big(\sum_{\ell=0}^{4} c_\ell(n)\, g_\ell\Big) +$$

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$$c_0(n)\, f + \cdots + c_4(n)\, S_n^4(f) = (S_k - 1)\left(\sum_{\ell=0}^{4} c_\ell(n)\, g_\ell\right) + \quad \overset{!}{=} 0$$

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Example. $$\underbrace{\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$+\begin{cases} c_0(n)\, f = (S_k - 1)\big(c_0(n)\, g_0\big) + c_0(n)\, \frac{nk}{(n+2k)^2+2} \\ c_1(n)\, S_n(f) = (S_k - 1)\big(c_1(n)\, g_1\big) + c_1(n)\, \frac{(n+1)k}{(n+2k+1)^2+2} \\ c_2(n)\, S_n^2(f) = (S_k - 1)\big(c_2(n)\, g_2\big) + c_2(n)\, \frac{(n+2)(k-1)}{(n+2k)^2+2} \\ c_3(n)\, S_n^3(f) = (S_k - 1)\big(c_3(n)\, g_3\big) + c_3(n)\, \frac{(n+3)(k-1)}{(n+2k+1)^2+2} \\ c_4(n)\, S_n^4(f) = (S_k - 1)\big(c_4(n)\, g_4\big) + c_4(n)\, \frac{(n+4)(k-2)}{(n+2k)^2+2} \end{cases}$$

$$c_0(n)\, f + \cdots + c_4(n)\, S_n^4(f) = (S_k - 1)\Big(\sum_{\ell=0}^{4} c_\ell(n)\, g_\ell\Big) + \quad \stackrel{!}{=} 0$$

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$$\begin{pmatrix} -4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} c_0(n) \\ c_1(n) \\ c_2(n) \\ c_3(n) \\ c_4(n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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A reduction-based approach (CHKL2015)

Example. $\underbrace{\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot S_n^2 + 1 \cdot S_n^4$

A reduction-based approach (CHKL2015)

Example. $$\underbrace{\frac{-n}{(nk+1)(nk+n+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$\begin{pmatrix} -4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot S_n^2 + 1 \cdot S_n^4$
- A certificate: $g = \frac{n+4}{n} \cdot g_0 + \frac{-2(n+4)}{n+2} \cdot g_2 + 1 \cdot g_4$

A reduction-based approach (CHKL2015)

Example. $$\underbrace{\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot S_n^2 + 1 \cdot S_n^4$

$$\frac{1}{nk+1} + \sum_{j=1}^{10} \frac{n(k+j)}{(n+2k+2j)^2+2}$$

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$$- \frac{(n+4)k}{(n+2k)^2+2} - \frac{2(n+4)k}{(n+2k+2)^2+2} - \frac{(n+4)k}{(n+2k+4)^2+2} - \frac{(n+4)(k+11)}{(n+2k+22)^2+2} + \frac{(n+4)(k+10)}{(n+2k+24)^2+2}$$

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- (−) May introduce superfluous terms in certificates

Integer-linear decompositions

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Definition. $p \in \mathbb{C}[n, k]$ irreducible, is integer-linear over $\mathbb{C}$ if

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- $P(z) \in \mathbb{C}[z]$ irreducible;
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integer-linear type

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Abramov-Le's criterion. $f \in \mathbb{C}(n, k)$ with $f = (S_k - 1)\left(\cdots\right) + \frac{a}{b}$.

$$f \text{ has a telescoper} \quad \Longleftrightarrow \quad b \text{ is integer-linear.}$$

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$$P_i(\lambda_i n + \mu_i k) \sim_{n,k} P_j(\lambda_j n + \mu_j k),\ i \neq j$$

$$\Updownarrow$$

$$(\lambda_i, \mu_i) = (\lambda_j, \mu_j)\ \&\ P_i(z) = P_j(z + \nu),\ \nu \in \mathbb{Z}$$

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- $P_i(z) \in \mathbb{C}[z]$ squarefree, $\gcd(P_i, P_i(z+\ell)) = 1$, $\forall\, \ell \in \mathbb{Z} \setminus \{0\}$;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
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Definition. $p \in \mathbb{C}[n,k]$ admits the integer-linear decomposition

$$p = P_0(n,k) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_0 \in \mathbb{C}[n,k]$ merely having non-integer-linear factors except for constants;
- $P_i(z) \in \mathbb{C}[z] \backslash \mathbb{C}$ squarefree, $\gcd(P_i, P_i(z+\ell)) = 1$, $\forall\, \ell \in \mathbb{Z} \backslash \{0\}$;
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$$\boxed{S_{\lambda,\mu}(P(\lambda n + \mu k)) = P(\lambda n + \mu k + 1)}$$

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Integer-linear operators of type (λ, μ)

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$$\sum_{i,j \in \mathbb{Z}} a_{ij} S_n^i S_k^j \quad \longmapsto \quad \sum_{i,j \in \mathbb{Z}} a_{ij} S_{\lambda,\mu}^{i\lambda + j\mu}$$

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$$M = (S_k - 1) \odot Q + R,$$

and either $R = 0$ or $0 \le \mathrm{ldeg}_{S_{\lambda,\mu}}(R) \le \deg_{S_{\lambda,\mu}}(R) < \mu - 1$.

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Q: $\mathrm{LSQ}(M, S_k - 1)$

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- $S_n(g) = S_{\lambda,\mu}^{\lambda}(g)$ and $S_k(g) = S_{\lambda,\mu}^{\mu}(g)$;
- $L(g) = \phi_{\lambda,\mu}(L)(g)$ for all $L \in \mathcal{A}$;

$$
\begin{aligned}
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Our new approach

Example. $\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}$

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$$f = (S_k - 1)\left(\frac{1}{nk+1}\right) + M\left(\frac{1}{(n+2k)^2+2}\right)$$

$$M = nk - n(k+1)S_{1,2}^{2} + n(k+11)S_{1,2}^{22}$$

Our new approach

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$$L(f) \;=\; L\cdot(S_k-1)\left(\frac{1}{nk+1}\right)+L\cdot M\left(\frac{1}{(n+2k)^2+2}\right)$$

$$L = c_0(n)+c_1(n)S_n+c_2(n)S_n^2+c_3(n)S_n^3+c_4(n)S_n^4$$

Our new approach

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$$L(f) \;=\; (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + (L \odot M)\left(\frac{1}{(n+2k)^2+2}\right)$$

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$$\begin{aligned} L(f) &= (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + (L \odot M)\left(\frac{1}{(n+2k)^2+2}\right) \\ &= (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + \left((S_k - 1) \odot Q + R\right)\left(\frac{1}{(n+2k)^2+2}\right) \end{aligned}$$

$$\begin{aligned} R &= \big(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\big) \\ &\quad + \big(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\big)S_n \end{aligned}$$

Our new approach

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$$L(f) = (S_k-1)\left(L\left(\frac{1}{nk+1}\right)\right)+(L\odot M)\left(\frac{1}{(n+2k)^2+2}\right)$$

$$= (S_k-1)\left(L\left(\frac{1}{nk+1}\right)+Q\left(\frac{1}{(n+2k)^2+2}\right)\right)+R\left(\frac{1}{(n+2k)^2+2}\right)$$

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$$\begin{cases} c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2) = 0 \\ c_1(n)(n+1)k + c_3(n)(n+3)(k-1) = 0 \end{cases}$$

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Recall: reduction-based approach

$$\begin{pmatrix} -4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Worst-case complexity (field operations)

Given $f \in \mathbb{C}(n, k)$ with $\deg_n(f) \le d_n$ and $\deg_k(f) \le d_k$.

RCT	NCT
$O^{\sim}(\mu^{\omega+2} d_n d_k^{\omega+3})$	$O^{\sim}(\mu^{\omega+1} d_n d_k^{\omega+2})$

- $\mu \in \mathbb{Z}^+$, $2 \le \omega \le 3$
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- Order of a minimal telescoper: $O(\mu d_k)$
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Timings (in seconds)

Test suite: $f(n,k) = (S_k - 1)\left(\frac{f_0(n,k)}{P_0(n,k)}\right) + \frac{a(n,k)}{P_1(2n+\mu k)\cdot P_2(4n+\mu k)}$.

- $P_i(z) = p_i(z)\cdot p_i(z+2^i)\cdot p_i(z+\mu)\cdot p_i(z+2^i+\mu)$,
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(d_1, d_2, μ)	RCT	NCT	Order
(1, 1, 1)	0.28	0.19	3
(1, 2, 1)	5.86	2.15	7
(1, 3, 1)	283.84	30.94	11
(1, 4, 1)	5734.80	448.09	15
(10, 2, 1)	7.79	3.18	7
(20, 2, 1)	9.49	4.21	7
(30, 2, 1)	16.57	10.17	8
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    - Avoids need to construct certificates
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- Future work.
  - Creative telescoping in extensive classes