Efficient Rational Creative Telescoping

Hui Huang

Symbolic Computation Group
University of Waterloo

Joint work with Mark Giesbrecht, George Labahn and Eugene Zima

Outline

- Technique of creative telescoping
- New algorithm for bivariate rational functions

The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n),\ldots,c_\rho(n)$ s.t.

$$c_0(n)f(n,k)+\cdots+c_\rho(n)f(n+\rho,k) \;=\; g(n,k+1)-g(n,k)$$

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Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n).$$

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Example. GIVEN $\binom{n}{k}$, FIND $\frac{k}{k-n-1}\binom{n}{k}$ and $-2,1$ s.t.

$$-2\binom{n}{k}+\binom{n+1}{k}=\frac{(k+1)}{(k+1)-n-1}\binom{n}{k+1}-\frac{k}{k-n-1}\binom{n}{k}$$

Then $F(n)=\sum_{k=0}^{n}\binom{n}{k}$ satisfies

$$-2F(n)+F(n+1)=0.$$

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$$c_0(n)f(n, k) + \cdots + c_\rho(n)f(n + \rho, k) \;=\; g(n, k + 1) - g(n, k)$$

Then $F(n) = \sum_{k=0}^{n} f(n, k)$ satisfies

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The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n), \ldots, c_\rho(n)$ s.t.

$$\left(c_0(n) + \cdots + c_\rho(n)\sigma_n^\rho\right)(f(n,k)) \;=\; (\sigma_k - 1)\left(g(n,k)\right)$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n).$$

Notation. $\sigma_n(f(n,k)) = f(n+1,k)$, $\sigma_k(f(n,k)) = f(n,k+1)$,
and $\Delta_k = \sigma_k - 1$.

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$$\underbrace{\boxed{c_0(n)+\cdots+c_\rho(n)\sigma_n^\rho}}_{\text{telescoper}}(f(n,k)) \;=\; \Delta_k\underbrace{\boxed{g(n,k)}}_{\text{certificate}}$$

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Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)

2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)

3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)

4. Hermite-like reduction based methods (since $\approx$ 2010)

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Reduction-based approach

Example. $\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}$

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$$f = \Delta_k\left(\boxed{g_0}\right) + \frac{nk}{(n+2k)^2+2}$$

$$\sum_{j=1}^{10} \frac{1}{n(k+j)+1} + \sum_{j=1}^{10} \frac{n(k+j)}{(n+2k+2j)^2+2}$$

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$$c_0(n)\, f = \Delta_k\Big(c_0(n)\, g_0\Big) + c_0(n)\, \frac{nk}{(n+2k)^2+2}$$

$$c_1(n)\, \sigma_n(f) = \Delta_k\Big(c_1(n)\, g_1\Big) + c_1(n)\, \frac{(n+1)k}{(n+2k+1)^2+2}$$

$$c_2(n)\, \sigma_n^2(f) = \Delta_k\Big(c_2(n)\, g_2\Big) + c_2(n)\, \frac{(n+2)(k-1)}{(n+2k)^2+2}$$

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Reduction-based approach

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$$

$$
c_0(n)\, f + \cdots + c_4(n)\, \sigma_n^4(f) = \Delta_k\left(\sum_{\ell=0}^{4} c_\ell(n)\, g_\ell\right) +
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$$c_0(n)\,f+\cdots+c_4(n)\,\sigma_n^4(f)=\Delta_k\Big(\sum_{\ell=0}^{4}c_\ell(n)\,g_\ell\Big)+\quad\overset{!}{=}0$$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

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$$= \sum_{j=1}^{10}\frac{1}{n(k+j)+1}+\frac{(n+4)(k+10)}{(n+2k+24)^2+2}-\frac{(n+4)(k+11)}{(n+2k+22)^2+2}-\frac{(n+4)k}{(n+2k+4)^2+2}-\frac{2(n+4)k}{(n+2k+2)^2+2}-\frac{(n+4)k}{(n+2k)^2+2}$$

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- Avoids need to construct certificates

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- Avoids need to construct certificates
- Can express certificates in symbolic sums

Reduction-based approach

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- ⊖ May introduce superfluous terms in certificates

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- g is expressed by a sparse form.

Integer-linear decomposition

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Definition. $p \in C[n,k]$ irreducible, is integer-linear over C if

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$$P_i(\lambda_i n + \mu_i k) \sim_{n,k} P_j(\lambda_j n + \mu_j k),\ i \neq j$$
$$\Updownarrow$$
$$(\lambda_i, \mu_i) = (\lambda_j, \mu_j)\ \&\ P_i(z) = P_j(z + \nu),\ \nu \in \mathbb{Z}$$

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- $P_i(z) \in C[z]$ squarefree, σ_z-free;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
- $e_{ij} \in \mathbb{Z}^+$; $0 = \nu_{i1} < \cdots < \nu_{in_i}$ in $\mathbb{Z}$;
- $P_i(\lambda_i n + \mu_i k) \nsim_{n,k} P_j(\lambda_j n + \mu_j k)$, $i \neq j$.

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Definition. $p \in C[n,k]$ admits the integer-linear decomposition

$$p = P_0(n,k) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_0 \in C[n,k]$ merely having non-integer-linear factors except for constants;
- $P_i(z) \in C[z]$ non-constant, squarefree, σ_z-free;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
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$$\boxed{L=c_0(n)+c_1(n)\sigma_n+c_2(n)\sigma_n^2+c_3(n)\sigma_n^3+c_4(n)\sigma_n^4}$$

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$$=\Delta_k\left(\cdots\right)+\left(\sum_{\ell=0}^{4}c_\ell(n)\,\sigma_n^{\ell}\right)\cdot(nk)\cdot\frac{1}{(n+2k)^2+2}$$

Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$L\cdot f=\Delta_k(L\cdot g_0)+L\cdot\underbrace{\left(n(k+11)\sigma_n^{22}-n(k+1)\sigma_n^{2}+nk\right)}_{M}\cdot\tfrac{1}{(n+2k)^2+2}$$

$$=\Delta_k(\cdots)+\Big(\sum_{\ell=0}^{4}c_\ell(n)(n+\ell)k\,\sigma_{(1,2)}^{\ell}\Big)\cdot\tfrac{1}{(n+2k)^2+2}$$

$$\boxed{\sigma_n\left(\tfrac{1}{(n+2k)^2+2}\right)=\sigma_{(1,2)}\left(\tfrac{1}{(n+2k)^2+2}\right)}$$

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$$=\Delta_k\big(\cdots\big)+\left((\sigma_k-1)\tilde{Q}+\tilde{R}\right)\cdot\frac{1}{(n+2k)^2+2}$$

$$\boxed{\sigma_k\left(\frac{1}{(n+2k)^2+2}\right)=\sigma_{(1,2)}^2\left(\frac{1}{(n+2k)^2+2}\right)}$$

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$$\left(c_0(n)nk+c_2(n)(n+2)(k-1)+c_4(n)(n+4)(k-2)\right)$$
$$+\left(c_1(n)(n+1)k+c_3(n)(n+3)(k-1)\right)\sigma_n$$

$$\boxed{\sigma_k\left(\frac{1}{(n+2k)^2+2}\right)=\sigma_{(1,2)}^2\left(\frac{1}{(n+2k)^2+2}\right)}$$

Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$L \cdot f = \Delta_k(L \cdot g_0) + L \cdot \underbrace{\left(n(k+11)\sigma_n^{22} - n(k+1)\sigma_n^2 + nk\right)}_{M} \cdot \frac{1}{(n+2k)^2+2}$$

$$= \Delta_k\left(\cdots\right) + \boxed{\tilde{R}} \cdot \frac{1}{(n+2k)^2+2}$$

$\big(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\big)$
$+ \big(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\big)\sigma_n$

Our new approach

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$$\left(c_0(n)nk+c_2(n)(n+2)(k-1)+c_4(n)(n+4)(k-2)\right)$$
$$+\left(c_1(n)(n+1)k+c_3(n)(n+3)(k-1)\right)\sigma_n$$

$$\boxed{L \text{ is a telescoper} \iff \tilde{R}=0}$$

Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

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$\left(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\right)$
$+ \left(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\right)\sigma_n$

$$\begin{cases} c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2) = 0 \\ c_1(n)(n+1)k + c_3(n)(n+3)(k-1) = 0 \end{cases}$$

Our new approach

Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$\begin{pmatrix} 0 & 0 & -n-2 & 0 & -2n-8 \\ n & 0 & n+2 & 0 & n+4 \\ 0 & 0 & 0 & -n-3 & 0 \\ 0 & n+1 & 0 & n+3 & 0 \end{pmatrix} \begin{pmatrix} c_0(n) \\ c_1(n) \\ c_2(n) \\ c_3(n) \\ c_4(n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Our new approach

Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

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Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$\begin{pmatrix} 0 & 0 & -n-2 & 0 & -2n-8 \\ n & 0 & n+2 & 0 & n+4 \\ 0 & 0 & 0 & -n-3 & 0 \\ 0 & n+1 & 0 & n+3 & 0 \end{pmatrix} \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

- A certificate: $g = L \cdot \boxed{g_0} + \operatorname{LeftQuot}(L \cdot M, \sigma_k - 1) \cdot \frac{1}{(n+2k)^2+2}$

  $\operatorname{LeftQuot}(\sigma_k^{10} - 1, \sigma_k - 1) \cdot \frac{1}{nk+1}$

Our new approach

Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

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Recall: reduction-based approach

$$\begin{pmatrix} 4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Outline of algorithm (iteration version)

Input. $f \in C(n, k)$.

Output. A minimal telescoper L and a certificate g when exist.

1 $\mathrm{den}(f) = P_0 \prod_{i,j} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$.

2 $f = \frac{f_0}{P_0} + \sum_{i,e} \overbrace{\boxed{\sum_{j=1}^{n_i} a_{ije} \sigma_{(\lambda_i,\mu_i)}^{\nu_{ij}}}}^{M_{ie}} \cdot \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

3 $\frac{f_0}{P_0} = \Delta_k(g) + r$. If $r \neq 0$, return "No telescoper exists!".

4 $M_{ie} = \Delta_k\big(\cdots\big) + R_{ie}$. If all $R_{ie} = 0$ then return $L = 1$ and $g = g + \sum\limits_{i,e} \mathrm{LeftQuot}(M_{ie}, \sigma_k - 1) \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

5 For $\rho = 1, 2, \ldots$ do

Find a telescoper L s.t. $L \cdot R_{ie} = \Delta_k\big(\cdots\big)$. If succeed return L and $g = L \cdot g + \sum\limits_{i,e} \mathrm{LeftQuot}(L \cdot M_{ie}, \sigma_k - 1) \cdot \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

Worst-case complexity (field operations)

Given $f \in C(n,k)$ with $\deg_n(f) \le d_n$ and $\deg_k(f) \le d_k$.

New_ub	New_it	RCT
$O^{\sim}(\mu^{\omega} d_n d_k^{\omega+1})$	$O^{\sim}(\mu^{\omega+1} d_n d_k^{\omega+2})$	$O^{\sim}(\mu^{\omega+2} d_n d_k^{\omega+3})$

- $\mu \in \mathbb{Z}^+$, $2 \le \omega \le 3$
- Without expanding the certificate
- Size of a minimal telescoper: $O(\mu^2 d_n d_k^3)$

Timings (in seconds)

Test suite: $f(n,k) = \Delta_k\left(\frac{f_0(n,k)}{P_0(n,k)}\right) + \frac{a(n,k)}{P_1(2n+\mu k)\cdot P_2(4n+\mu k)}$.

- $P_i(z) = p_i(z) \cdot p_i(z+2^i) \cdot p_i(z+\mu) \cdot p_i(z+2^i+\mu)$,
- $\mu \in \mathbb{Z}$, $\deg_{n,k}(a) = d_1$, $\deg_{n,k}(P_0) = \deg_z(p_i) = d_2$.

(d_1, d_2, μ)	RCT	New_ub	New_it	Order	Upper
(1, 1, 1)	0.28	0.19	0.19	3	4
(1, 2, 1)	5.86	4.88	2.15	7	8
(1, 3, 1)	283.84	630.61	30.94	11	12
(1, 4, 1)	5734.80	37272.09	448.09	15	16
(10, 2, 1)	7.79	11.89	3.18	7	8
(20, 2, 1)	9.49	25.22	4.21	7	8
(30, 2, 1)	16.57	9.67	10.17	8	8
(30, 2, 3)	807.31	39.37	41.16	12	12
(30, 2, 5)	4875.63	305.16	344.81	20	20
(30, 2, 7)	34430.03	1479.36	1240.54	28	28

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Summary

Result.

- A creative telescoping algorithm for bivariate rational function
  - Avoids need to construct certificates
  - Expresses certificates in precise and manipulable sparse forms
  - Has better control in size of intermediate expression
  - Easier to analyze, and more efficient

Future work.

- Generalize to hypergeometric terms