Order-Degree-Height Surfaces for Linear Operators

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Joint work with Manuel Kauers and Gargi Mukherjee

Outline

- Linear operators and their size
- Order-degree-height surfaces
  - Left common multiples
  - Creative telescoping
  - Contraction ideals

D-finite sequences

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Definition. A sequence $(f(n))_{n\in\mathbb{N}}$ is called D-finite if there exist polynomials $p_0(n),\ldots,p_r(n)$, not all zero, such that

$$p_0(n)f(n)+p_1(n)f(n+1)+\cdots+p_r(n)f(n+r) \;=\; 0.$$

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Example. Consider

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}\binom{2n}{2k}.$$

Then

$$\begin{aligned}(-48n^3 - 152n^2 - 144n - 40)\, f(n) \\ + (-42n^3 - 154n^2 - 178n - 64)\, f(n+1) \\ + (6n^3 + 25n^2 + 32n + 12)\, f(n+2) = 0\end{aligned}$$

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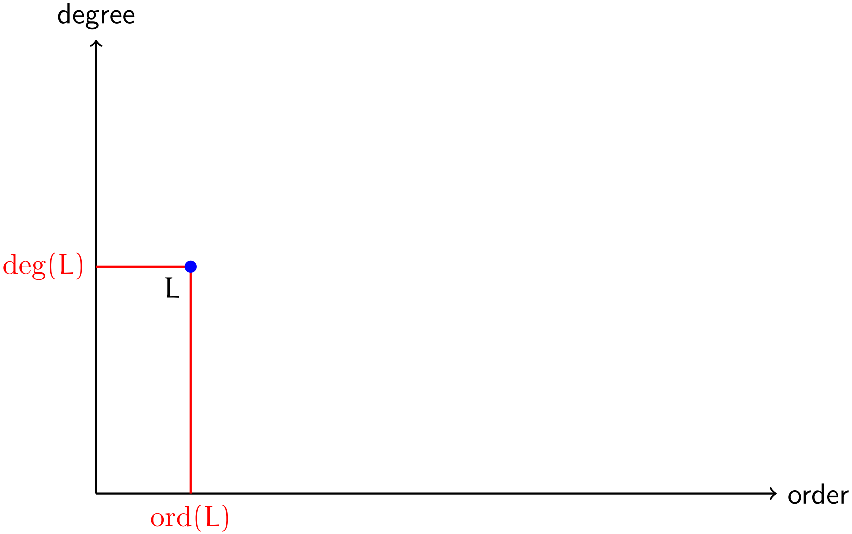
Order-degree curve



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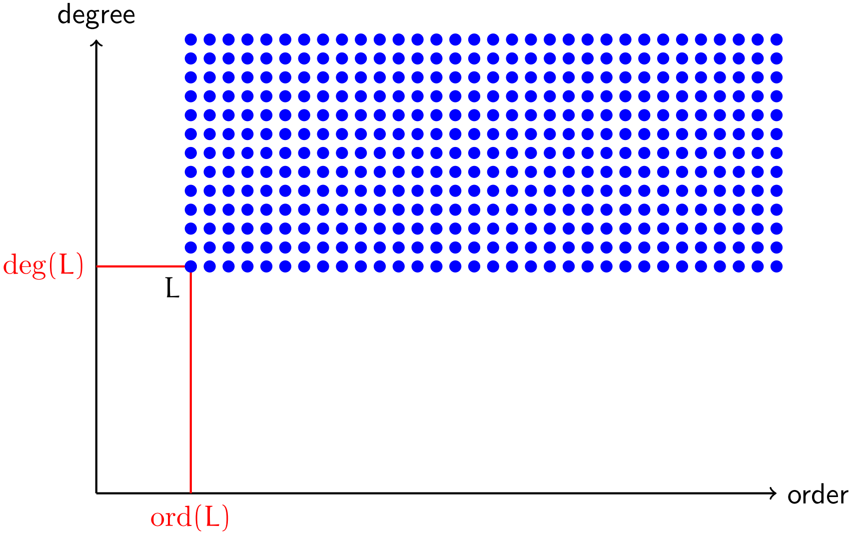

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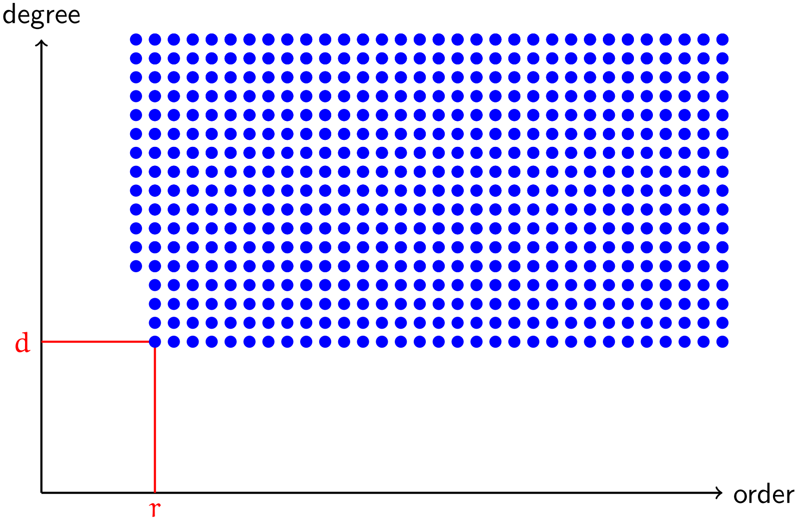

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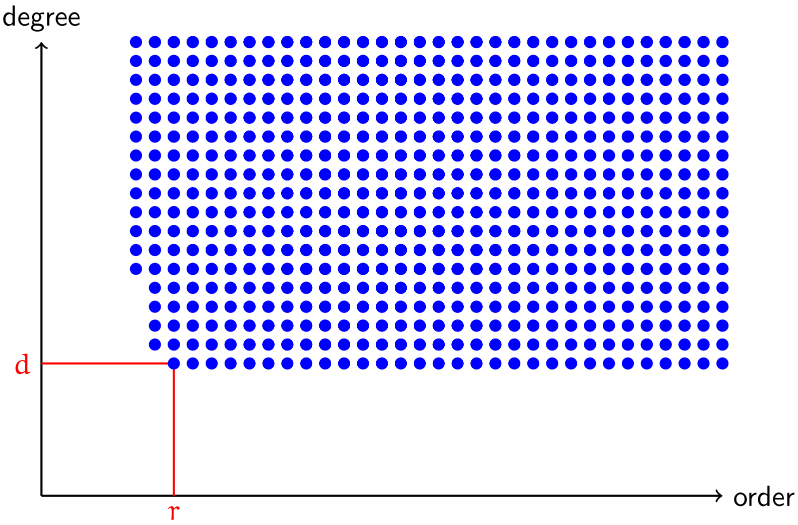

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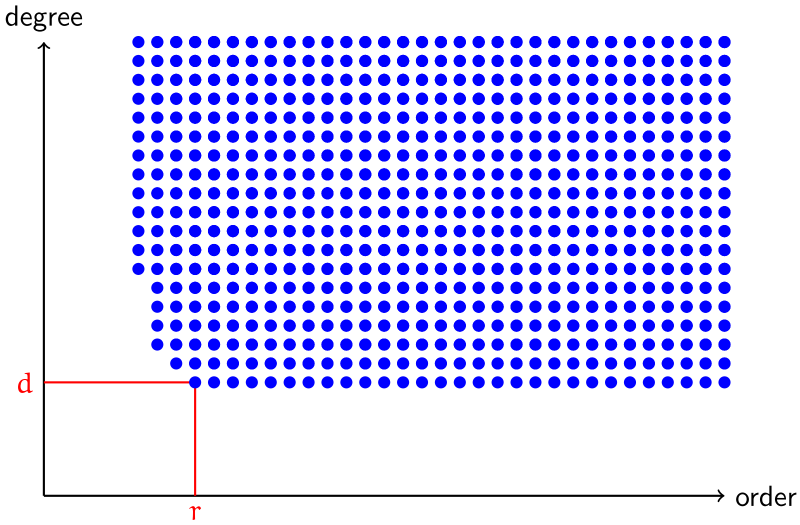

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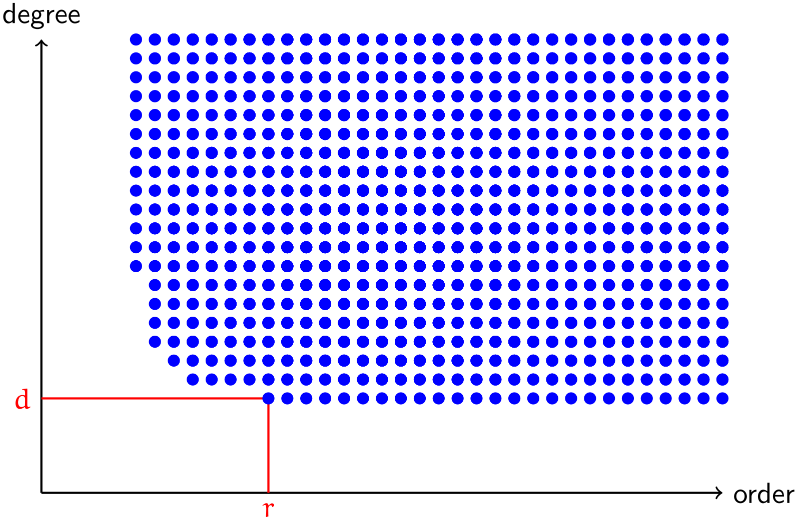

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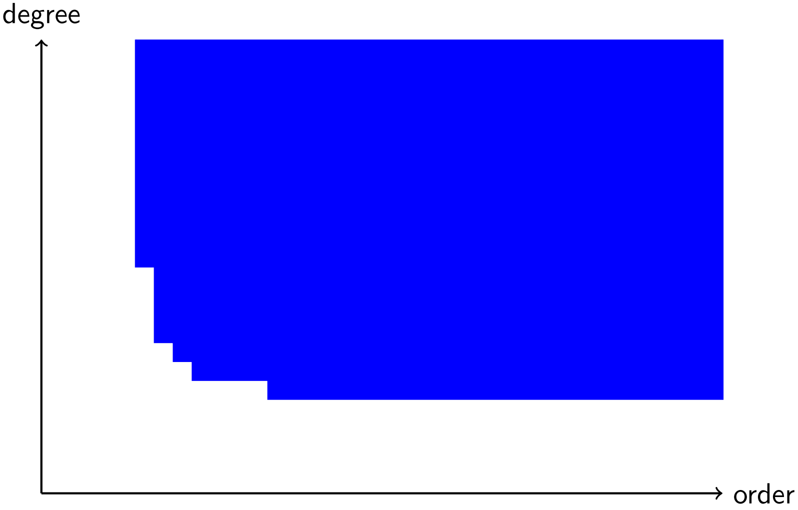

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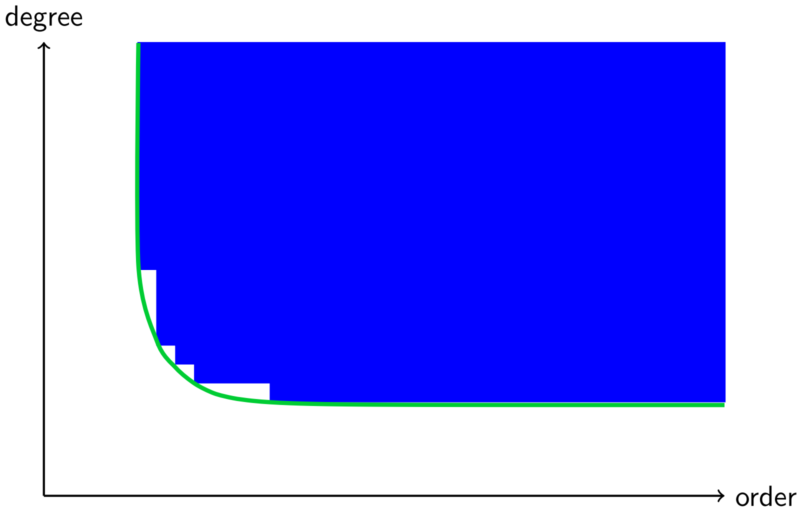

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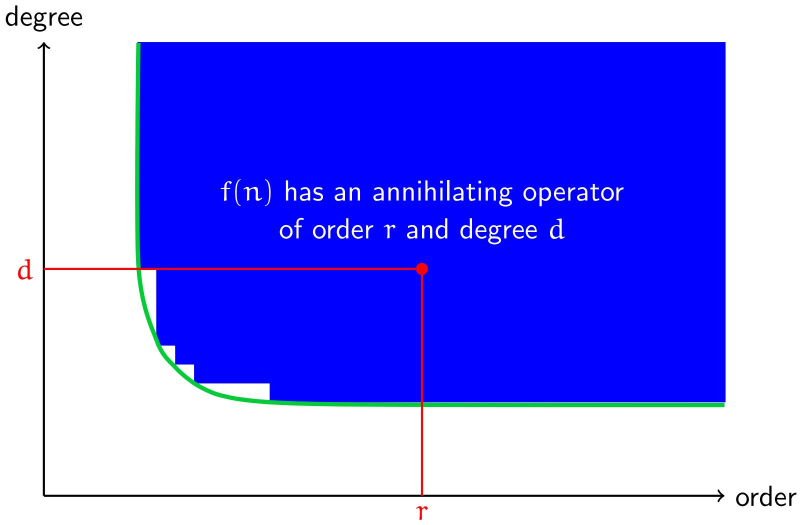

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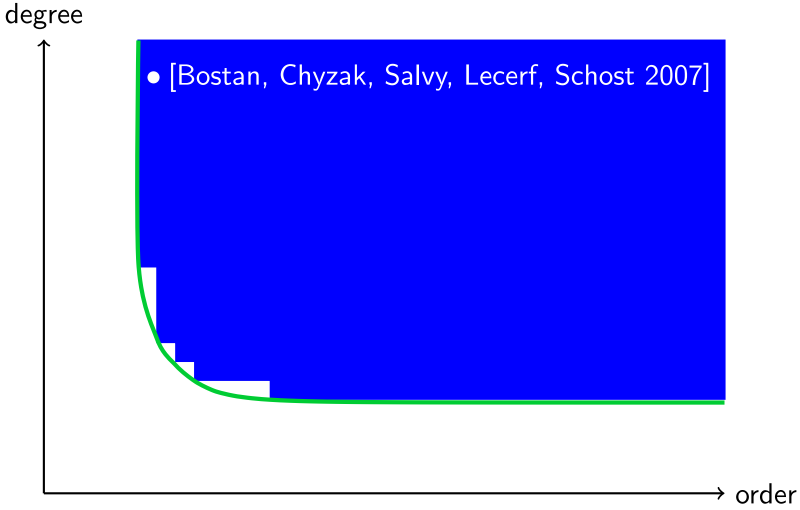

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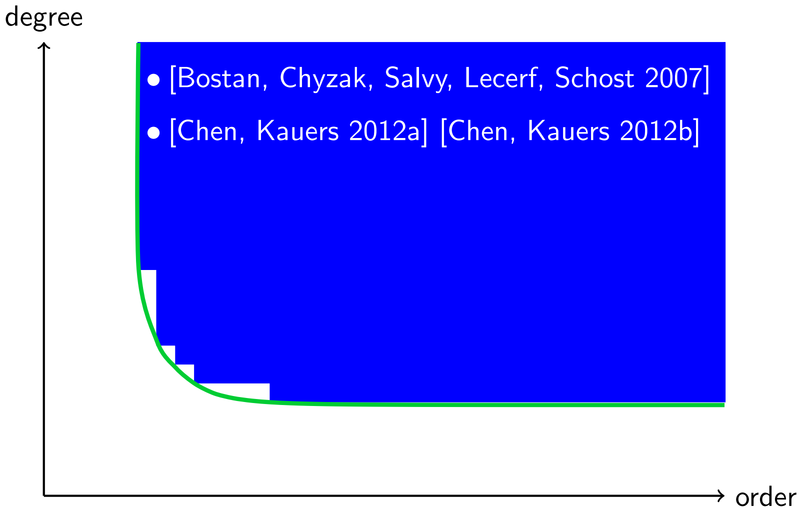

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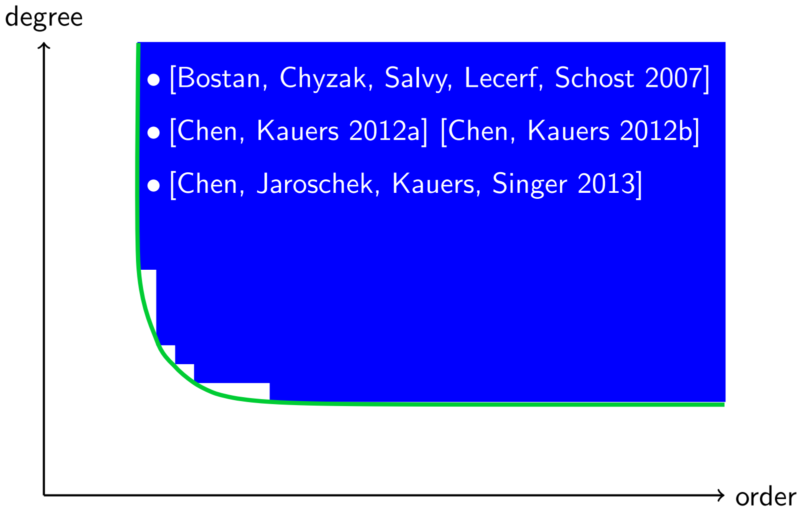

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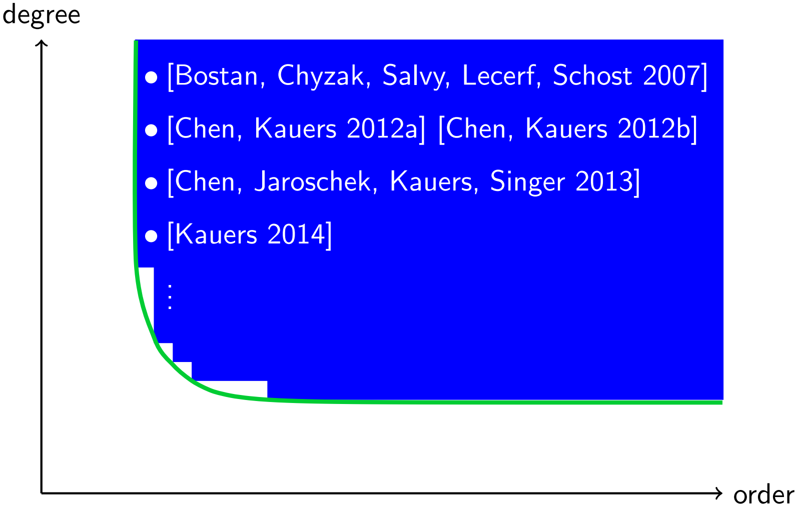

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Motivating example

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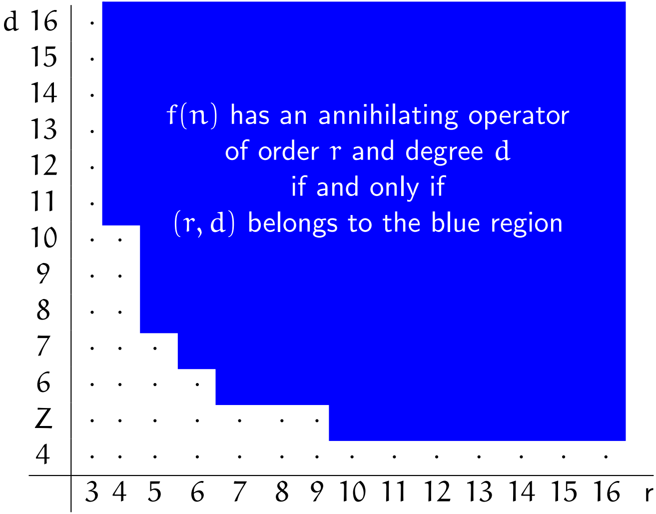

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d \ r	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	·	8	7	7	6	7	7	7	7	8	8	8	9	8
15	·	8	7	7	7	7	7	7	7	7	8	8	8	8
14	·	8	7	7	7	6	7	7	7	7	8	8	8	8
13	·	8	8	7	7	6	7	7	7	7	7	8	8	8
12	·	8	8	7	7	7	7	7	7	7	7	7	8	8
11	·	9	8	7	7	7	7	7	7	7	7	7	7	7
10	·	·	9	7	7	7	7	7	7	7	7	7	7	7
9	·	·	11	8	7	7	7	7	7	7	7	7	7	7
8	·	·	14	9	8	7	7	7	7	7	7	7	7	7
7	·	·	·	11	9	8	8	7	7	7	7	8	7	7
6	·	·	·	·	13	10	9	8	8	8	8	8	8	8
Z	·	·	·	·	·	·	·	8	8	8	8	8	8	8
4	·	·	·	·	·	·	·	·	·	·	·	·	·	·

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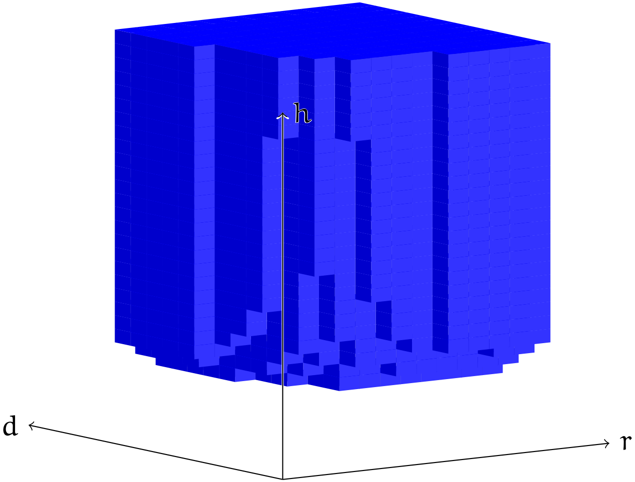

Our set-up

Consider $L \in C[t][n][S_n]$, where

- C is a field of characteristic zero,
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L = {} & (2n^3 + (3t+8)n^2 + (t^2+9t+11)n + 2t^2 + 7t + 5) \\
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Theorem. [Kauers 2014] For any $r, d \in \mathbb{N}$ with

$$r \geq \sum_{\ell=1}^{m} r_\ell \quad \text{and} \quad d \geq \frac{(r+1)\sum_{\ell=1}^{m} d_\ell - \sum_{\ell=1}^{m} r_\ell d_\ell}{r + 1 - \sum_{\ell=1}^{m} r_\ell},$$

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$$\sum_{i=0}^{r-r_\ell} \sum_{j=0}^{d-d_\ell} \sum_{k=0}^{h-h_\ell} c_{ijk\ell} t^k n^j S_n^i \text{ with } c_{ijk\ell} \in C \text{ to be determined}$$

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&(r+1)(d+1)(h+1) - (r+1)(d+1)\sum_{\ell=1}^{m} h_\ell \ + \ (h+1)\sum_{\ell=1}^{m} r_\ell d_\ell \\
&- (r+1)(h+1)\sum_{\ell=1}^{m} d_\ell \ + \ (d+1)\sum_{\ell=1}^{m} r_\ell h_\ell \\
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Example

Consider randomly chosen $L_1, L_2 \in \mathbb{Q}[t][n][S_n]$ with

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d \ r	0	1	2	3	4	5	6	7	8	9	10
10	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$15\mid 5$	$6\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 2$	$3\mid 2$
9	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$21\mid 5$	$6\mid 5$	$4\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 2$
8	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$37\mid 5$	$7\mid 5$	$5\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$
7	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$9\mid 6$	$5\mid 4$	$4\mid 3$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$
6	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$12\mid 8$	$7\mid 5$	$5\mid 4$	$4\mid 3$	$4\mid 3$	$4\mid 3$	$3\mid 3$
5	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$31\mid 19$	$10\mid 7$	$7\mid 5$	$5\mid 4$	$5\mid 4$	$4\mid 4$	$4\mid 3$
4	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$31\mid 19$	$12\mid 8$	$9\mid 6$	$7\mid 5$	$6\mid 5$	$6\mid 4$
3	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$37\mid 5$	$21\mid 5$	$15\mid 5$
2	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$
1	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$
0	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$

Example

Consider randomly chosen $L_1, L_2 \in \mathbb{Q}[t][n][S_n]$ with

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- $\operatorname{ord}(L_2) = 1, \deg(L_2) = 2, \operatorname{ht}(L_2) = 1.$

d \ r	0	1	2	3	4	5	6	7	8	9	10
10	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$15 \mid 5$	$6 \mid 4$	$4 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 2$	$3 \mid 2$
9	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$21 \mid 5$	$6 \mid 5$	$4 \mid 4$	$4 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 2$
8	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$37 \mid 5$	$7 \mid 5$	$5 \mid 4$	$4 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 3$
7	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$9 \mid 6$	$5 \mid 4$	$4 \mid 3$	$4 \mid 3$	$3 \mid 3$	$3 \mid 3$	$3 \mid 3$
6	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$12 \mid 8$	$7 \mid 5$	$5 \mid 4$	$4 \mid 3$	$4 \mid 3$	$4 \mid 3$	$3 \mid 3$
5	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$31 \mid 19$	$10 \mid 7$	$7 \mid 5$	$5 \mid 4$	$5 \mid 4$	$4 \mid 4$	$4 \mid 3$
4	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$31 \mid 19$	$12 \mid 8$	$9 \mid 6$	$7 \mid 5$	$6 \mid 5$	$6 \mid 4$
3	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$37 \mid 5$	$21 \mid 5$	$15 \mid 5$
2	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$
1	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$
0	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$

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d \ r	0	1	2	3	4	5	6	7	8	9	10
10	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$15\mid 5$	$6\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 2$	$3\mid 2$
9	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$21\mid 5$	$6\mid 5$	$4\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 2$
8	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$37\mid 5$	$7\mid 5$	$5\mid 4$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$
7	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$9\mid 6$	$5\mid 4$	$4\mid 3$	$4\mid 3$	$3\mid 3$	$3\mid 3$	$3\mid 3$
6	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$12\mid 8$	$7\mid 5$	$5\mid 4$	$4\mid 3$	$4\mid 3$	$4\mid 3$	$3\mid 3$
5	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$31\mid 19$	$10\mid 7$	$7\mid 5$	$5\mid 4$	$5\mid 4$	$4\mid 4$	$4\mid 3$
4	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$31\mid 19$	$12\mid 8$	$9\mid 6$	$7\mid 5$	$6\mid 5$	$6\mid 4$
3	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$37\mid 5$	$21\mid 5$	$15\mid 5$
2	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$
1	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$
0	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$

Creative telescoping

Given a (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)x^n y^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')},$$

where $c \in C[t][n,k], x, y \in C[t], a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i' \in \mathbb{N}$ and $a_i'', b_i'', u_i'', v_i'' \in C[t]$.

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Fact. There are $p_0, \ldots, p_r \in C[t][n]$ and $Q \in C(t)(n,k)$ such that

$$(p_0(n) + p_1(n)S_n + \cdots + p_r(n)S_n^r) \cdot f(n,k) = (S_k - 1)Q(n,k) \cdot f(n,k).$$

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$$r \geq \nu \quad \text{and} \quad d > \frac{(\mu\nu - 1)r + \frac{1}{2}\nu(2\delta + |\lambda| + 3 - (1 + |\lambda|)\nu) - 1}{r - \nu + 1},$$

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Theorem. For any $r, d, h \in \mathbb{N}$ with

$$(r+1)(d+1)(h+1) - (r\mu + d + 1 + \vartheta_n)(r\mu + \vartheta_k + 1)\eta$$
$$- (2r\mu + d + 2 + \vartheta_n + \vartheta_k - \nu)(r\xi + h + 1 + \vartheta_t - \eta)\nu > 0,$$

there exists a telescoper for $f(n,k)$ of order r, degree d, height h.

Remark. $\vartheta_n, \vartheta_t, \vartheta_z, \mu, \nu, \xi, \eta \in \mathbb{N}$ merely depending on $f(n,k)$.

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Example

Consider

$$f(n,k) = k\frac{\Gamma(n+k+t^2)}{\Gamma(n-k+t)}.$$

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<table>
<tr><td>d 10</td><td>·|·</td><td>·|·</td><td>62|9</td><td>31|7</td><td>27|6</td><td>26|5</td><td>26|5</td><td>27|5</td><td>28|5</td><td>29|4</td><td></td></tr>
<tr><td>9</td><td>·|·</td><td>·|·</td><td>86|9</td><td>36|7</td><td>30|6</td><td>29|5</td><td>30|5</td><td>30|5</td><td>32|5</td><td>33|5</td><td></td></tr>
<tr><td>8</td><td>·|·</td><td>·|·</td><td>158|9</td><td>45|7</td><td>36|6</td><td>35|5</td><td>35|5</td><td>36|5</td><td>37|5</td><td>39|5</td><td></td></tr>
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<tr><td>6</td><td>·|·</td><td>·|·</td><td>·|9</td><td>114|7</td><td>69|6</td><td>61|6</td><td>59|5</td><td>60|5</td><td>61|5</td><td>63|5</td><td></td></tr>
<tr><td>5</td><td>·|·</td><td>·|·</td><td>·|9</td><td>·|7</td><td>160|7</td><td>113|6</td><td>102|5</td><td>98|5</td><td>99|5</td><td>101|5</td><td></td></tr>
<tr><td>4</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|7</td><td>·|7</td><td>·|6</td><td>570|6</td><td>371|5</td><td>312|5</td><td>288|5</td><td></td></tr>
<tr><td>3</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|10</td><td>·|8</td><td>·|6</td><td>·|6</td><td>·|6</td><td>·|6</td><td>·|6</td><td></td></tr>
<tr><td>2</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|8</td><td>·|8</td><td>·|8</td><td>·|8</td><td>·|8</td><td></td></tr>
<tr><td>1</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td></td></tr>
<tr><td>0</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td>·|·</td><td></td></tr>
<tr><td></td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>r</td></tr>
</table>

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d \ r	0	1	2	3	4	5	6	7	8	9
10	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$62\mid 9$	$31\mid 7$	$27\mid 6$	$26\mid 5$	$26\mid 5$	$\mathbf{27}\mid 5$	$28\mid 5$	$29\mid 4$
9	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$86\mid 9$	$36\mid 7$	$30\mid 6$	$29\mid 5$	$30\mid 5$	$\mathbf{30}\mid 5$	$32\mid 5$	$33\mid 5$
8	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$158\mid 9$	$45\mid 7$	$36\mid 6$	$35\mid 5$	$35\mid 5$	$\mathbf{36}\mid 5$	$37\mid 5$	$39\mid 5$
7	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 9$	$62\mid 7$	$47\mid 6$	$43\mid 6$	$43\mid 5$	$\mathbf{44}\mid 5$	$46\mid 5$	$47\mid 5$
6	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 9$	$114\mid 7$	$69\mid 6$	$61\mid 6$	$59\mid 5$	$\mathbf{60}\mid 5$	$61\mid 5$	$63\mid 5$
5	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 9$	$\cdot\mid 7$	$160\mid 7$	$113\mid 6$	$102\mid 5$	$98\mid 5$	$\mathbf{99}\mid 5$	$101\mid 5$
4	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 7$	$\cdot\mid 7$	$\cdot\mid 6$	$570\mid 6$	$371\mid 5$	$312\mid 5$	$288\mid 5$
3	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 10$	$\cdot\mid 8$	$\cdot\mid 6$	$\cdot\mid 6$	$\cdot\mid 6$	$\cdot\mid 6$	$\cdot\mid 6$
2	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid 8$	$\cdot\mid 8$	$\cdot\mid 8$	$\cdot\mid 8$	$\cdot\mid 8$
1	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$
0	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$	$\cdot\mid\cdot$

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Given an operator $L \in C[t][n][S_n]$.

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How does $L_1 \in \mathrm{Con}\langle L \rangle$ give rise to the size of elements of $\mathrm{Con}\langle L \rangle$?

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Theorem. [Chen, Jaroschek, Kauers, Singer 2013]
Let $L_1 \in \mathrm{Con}\langle L \rangle$, $p \in C[t][n]$ and $P \in C[t][n][S_n]$ with

$$pL_1 = PL \quad \text{and} \quad \deg_n(p) > \deg_n(\mathrm{lc}_{S_n}(P)).$$

Then for any $r, d \in \mathbb{N}$ with $r \geq \mathrm{ord}(L)$ and

$$d \geq \deg_n(L) - \left(1 - \frac{\mathrm{ord}(L_1) - \mathrm{ord}(L)}{r + 1 - \mathrm{ord}(L)}\right)\left(\deg_n(p) - \deg_n(\mathrm{lc}_{S_n}(P))\right),$$

there exists $Q \in C(t)(n)[S_n]$ such that $QL \in C[t][n][S_n]$ has order r and degree d.

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Then for any $r, d, h \in \mathbb{N}$ with $r \geq \mathrm{ord}(L), \mathrm{ord}(L_1)$ and

$$
\begin{aligned}
&(r - \mathrm{ord}(L) + 1)\Big(-(r \deg_n(p) - \xi_n + \eta_n)(r \deg_t(p) - \xi_t + \eta_t) \\
&+ (r \deg_n(p) + d - \deg(L) + 1 - \xi_n)(r \deg_t(p) + h - \mathrm{ht}(L) + 1 - \xi_t) \\
&+ \eta_n\eta_t + \lambda_n\lambda_t\Big) - \mathrm{ord}(P)\lambda_n\lambda_t > 0,
\end{aligned}
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Remark. $\lambda_n, \lambda_t, \xi_n, \xi_t, \eta_n, \eta_t \in \mathbb{N}$ merely depending on L, L_1, p, P.

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Example

- L is a minimal telescoper for $k\Gamma(n+k+t^2)/\Gamma(n-k+t)$ with $\mathrm{ord}(L)=2$, $\deg(L)=5$, $\mathrm{ht}(L)=9$ and
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d \ r	1	2	3	4	5	6	7
9	·|·	·|9	8|7	8|6	8|5	8|5	8|5
8	·|·	·|9	8|7	8|6	8|5	8|5	8|5
7	·|·	·|9	10|7	**12**|6	13|6	15|5	17|5
6	·|·	·|9	13|7	**18**|6	23|6	28|5	33|5
5	·|·	·|9	23|7	**38**|7	53|6	68|5	83|5
4	·|·	·|·	·|7	·|7	·|6	·|6	·|5
3	·|·	·|·	·|10	·|8	·|6	·|6	·|6
2	·|·	·|·	·|·	·|·	·|8	·|8	·|8
1	·|·	·|·	·|·	·|·	·|·	·|·	·|·

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7	$\cdot \mid \cdot$	$\cdot \mid 9$	$10 \mid 7$	**12** $\mid 6$	$13 \mid 6$	$15 \mid 5$	$17 \mid 5$
6	$\cdot \mid \cdot$	$\cdot \mid 9$	$13 \mid 7$	**18** $\mid 6$	$23 \mid 6$	$28 \mid 5$	$33 \mid 5$
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4	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid 7$	$\cdot \mid 7$	$\cdot \mid 6$	$\cdot \mid 6$	$\cdot \mid 5$
3	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid 10$	$\cdot \mid 8$	$\cdot \mid 6$	$\cdot \mid 6$	$\cdot \mid 6$
2	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid 8$	$\cdot \mid 8$	$\cdot \mid 8$
1	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$	$\cdot \mid \cdot$

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- Understand and eliminate the quadratic term in the order.
- Order-degree-height surfaces for operators in $\mathbb{Z}[x][\partial_x]$.