Two Applications of the Modified Abramov-Petkovšek Reduction

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Outline

- Modified Abramov–Petkovšek reduction
- Reduction-based creative telescoping
- Upper and lower order bounds for telescopers

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Notation. For $f(y)$,

$$\sigma_y(f) := f(y+1) \quad \text{and} \quad \Delta_y(f) := f(y+1) - f(y).$$

Hypergeometric terms

Definition. A nonzero term $T(y)$ is hypergeometric over $\mathbb{C}(y)$ if

$$\frac{\sigma_y(T)}{T} \in \mathbb{C}(y).$$

Example. $T \in \mathbb{C}(y) \setminus \{0\}$, $y!$, $\binom{y}{4}$, ...

Definition. A hypergeom. term $T(y)$ is summable if

$$T(y) = \Delta_y(\text{hypergeom.}).$$

Example. $y \cdot y! = (y+1)! - y!$ is summable; but $y!$ is not.

Multiplicative decomposition

Definition. $u/v \in \mathbb{C}(y)$ is shift-reduced if

$$\forall \ell \in \mathbb{Z}, \quad \gcd\left(v, \sigma_y^{\ell}(u)\right) = 1.$$

For a hypergeom. term $T(y)$, $\exists\ K, S \in \mathbb{C}(y)$ with K shift-reduced s.t.

$$T = SH, \quad \text{where } \frac{\sigma_y(H)}{H} = K.$$

Call

- K, a kernel of T
- S, the corresponding shell of T

Modified A.-P. reduction (CHKL2015)

Theorem. Let $T(y)$ be hypergeom. with multi. decomp.

$$T = SH \quad \text{with} \quad \frac{u}{v} := \frac{\sigma_y(H)}{H}.$$

Then $\exists\ a, b, q \in \mathbb{C}[y]$ with $\deg_y(a) < \deg_y(b)$ s.t.

$$T = \Delta_y(\dots) + \left(\frac{a}{b} + \frac{q}{v}\right) H,$$

Moreover,

$$T \text{ is summable } \iff a = q = 0.$$

Term bound for q

Notation. (Iverson bracket)

$$[\![\cdots]\!] = \begin{cases} 1 & \cdots \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Proposition.

$$\#\text{ terms of } q \quad \leq \quad \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]. \end{array}$$

Bivariate hypergeometric terms

Definition. A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if

$$\frac{\sigma_x(T)}{T}, \ \frac{\sigma_y(T)}{T} \ \in \mathbb{C}(x, y).$$

Creative-telescoping problem. Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in C(x)\langle \sigma_x \rangle$ s.t.

$$L(T) = \Delta_y(G) \text{ for some hypergeom. } G(x, y)$$

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↙ telescoper

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telescoper (L) certificate (G)

Existence of telescopers

Definition. $p \in \mathbb{C}[x, y]$ is integer-linear if

$$p = \prod_i (\alpha_i x + \beta_i y + \gamma_i)$$

where $\alpha_i, \beta_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{C}$.

Existence criterion (Wilf&Zeilberger1992, Abramov2003).

In the initial reduction

$$T = \Delta_y(\ldots) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H.$$

Then

$$T \text{ has a telescoper } \Leftrightarrow b_0 \text{ is integer-linear.}$$

Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

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$$\sigma_x(T) = \Delta_y\Big(\cdots\Big) + \left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H$$

$$\vdots$$

$$\sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + \left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H$$

Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

Idea. Set $T = SH$, a multi. decomp.

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H$$

$$c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, \left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H$$

$$\vdots$$

$$c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, \left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H$$

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Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

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$$
+\left\{
\begin{aligned}
c_0(x)\, T &= \Delta_y\Big(\cdots\Big) + c_0(x)\,\left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \\
c_1(x)\, \sigma_x(T) &= \Delta_y\Big(\cdots\Big) + c_1(x)\,\left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H \\
&\vdots \\
c_\rho(x)\, \sigma_x^\rho(T) &= \Delta_y\Big(\cdots\Big) + c_\rho(x)\,\left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H
\end{aligned}
\right.
$$

$$
\left(\sum_{i=0}^{\rho} c_i(x)\sigma_x^i\right)(T) = \Delta_y\Big(\cdots\Big) + \left(\sum_{j=0}^{\rho} c_j(x)\left(\frac{a_j}{b_j} + \frac{q_j}{v}\right)\right) H
$$

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Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

Idea. Set $T = SH$, a multi. decomp.

$$\sum_{i=0}^{\rho} c_i(x)\sigma_x^i \text{ is a telescoper for } T$$

$$\Updownarrow$$

$$\sum_{j=0}^{\rho} c_j(x)\left(\frac{a_j}{b_j} + \frac{q_j}{v}\right) = 0$$

$$\Updownarrow$$

$$\begin{cases} c_0(x)\frac{a_0(x,y)}{b_0(x,y)} + \cdots + c_\rho(x)\frac{a_\rho(x,y)}{b_\rho(x,y)} = 0 \\ c_0(x)q_0(x,y) + \cdots + c_\rho(x)q_\rho(x,y) = 0 \end{cases}$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

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- A multi. decomp. $T = SH$ where

$$S = \frac{1}{x+2y}$$

  and

$$H = y! \quad \text{with} \quad \frac{\sigma_y(H)}{H} = y+1.$$

- $u := y+1$ and $v := 1$.

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right) H$$

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$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

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$$\left(\frac{2}{x+2y} + \frac{0}{v}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{v}\right)$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x) \cdot \left(\frac{2}{x+2y} + \frac{0}{v} \right)$$

$$+\, c_1(x) \cdot \left(\frac{2}{x+2y+1} + \frac{0}{v} \right)$$

$$= 0$$

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No solution in $\mathbb{C}(x)$!

$$= 0$$

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$$\sigma_x^2(T) = \Delta_y\left(g_2\right) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)H$$

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Example

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$$T = \frac{1}{x+2y} \cdot y!$$

$$\left(\frac{2}{x+2y} + \frac{0}{v}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{v}\right)$$

$$\left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)$$

$$\left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v}\right)$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x) \cdot \left(\frac{2}{x+2y} + \frac{0}{v} \right)$$

$$+ c_1(x) \cdot \left(\frac{2}{x+2y+1} + \frac{0}{v} \right)$$

$$+ c_2(x) \cdot \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v} \right)$$

$$+ c_3(x) \cdot \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v} \right)$$

$$= 0$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\begin{aligned}
&- 2\cdot \left(\frac{2}{x+2y} + \frac{0}{v} \right) \\
&+ 2\cdot \left(\frac{2}{x+2y+1} + \frac{0}{v} \right) \\
&- x\cdot \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v} \right) \\
&+ (x+1)\cdot \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v} \right) \\
&= 0
\end{aligned}$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

Therefore,

- the minimal telescoper for T w.r.t. y is

$$L = (x+1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

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- the corresponding certificate is

$$\begin{aligned} G &= (x+1) \cdot g_3 - x \cdot g_2 + 2 \cdot g_1 - 2 \cdot g_0 \\ &= \frac{2y!}{(x+2y)(x+2y+1)} \end{aligned}$$

Upper bound

Example (cont.). $T = y!/(x + 2y)$.

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- Initial reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

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$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$c_0(x)\left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) + \cdots + c_\rho(x)\left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) = 0$$

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- Property. $b_i = x + 2y$ or $x + 2y + 1$.

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{vars} = \rho + 1 \qquad \begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

Upper bound

Example (cont.). $T = y!/(x + 2y)$.

$\Uparrow$ Prop. $b_i = x + 2y$ or $x + 2y + 1$

$\#\text{vars} = \rho + 1$

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Upper bound

Example (cont.). $T = y!/(x+2y)$.

common denom. $B = (x+2y)(x+2y+1)$

$\Uparrow$ Prop. $b_i = x+2y$ or $x+2y+1$

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$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\Uparrow \quad \deg_y(a_i) < \deg_y(b_i)$$

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Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) = 2$$

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$$\text{common denom. } B = (x+2y)(x+2y+1)$$

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Upper bound

Example (cont.). $T = y!/(x+2y)$.

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$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

$$\Downarrow \quad \text{Prop. } \# \text{ terms of } q \leq \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \end{array}$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) = 2$$

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$$\#\text{eqns over } \mathbb{C}(x) = 1$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) = 2$$

$$\Uparrow \ \deg_y(a_i) < \deg_y(b_i)$$

$$\text{common denom. } B = (x+2y)(x+2y+1)$$

$$\Uparrow \ \text{Prop. } b_i = x+2y \text{ or } x+2y+1$$

$$\#\text{vars} = \rho + 1 \qquad \#\text{eqns over } \mathbb{C}(x) = 3 \qquad \begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

$$\Downarrow \ \text{Prop. } \#\text{ terms of } q \leq \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \end{array}$$

$$\#\text{eqns over } \mathbb{C}(x) = 1$$

Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) = 2$$

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$$\begin{array}{l} \#\text{vars} = \rho+1 \\ \#\text{eqns over } \mathbb{C}(x) = 3 \end{array} \quad \begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

$$\Downarrow \ \text{Prop. } \#\text{ terms of } q \leq \begin{array}{l} \max\left(\deg_y(u),\ \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \end{array}$$

$$\#\text{eqns over } \mathbb{C}(x) = 1$$

Conclusion. Upper bound is 3.

New upper bound

Theorem. Assume T has initial reduction

$$T = \Delta_y(\cdots) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H$$

with $b_0 = c \prod_{i=1}^{m} \prod_{k=0}^{d_i} (\alpha_i x + \beta_i y + \gamma_i + k)^{c_{ik}}$, and for each $i \neq j$, either

$$\alpha_i \neq \alpha_j \quad \text{or} \quad \beta_i \neq \beta_j \quad \text{or} \quad \gamma_i - \gamma_j \notin \mathbb{Z}.$$

Then the order of the minimal telescoper for T w.r.t. y is no more than

$$B_{New} := \max\{\deg_y(u), \deg_y(v)\} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] + \sum_{i=1}^{m} \beta_i \cdot \max_{0 \leq k \leq d_i} \{c_{ik}\}$$

Apagodu-Zeilberger upper bound (2006)

Definition. A hypergeom. term T is said to be proper if it is of the form

$$T = p(x,y)\prod_{i=1}^{m}\frac{(\alpha_i x+\alpha_i' y+\alpha_i'')!(\beta_i x-\beta_i' y+\beta_i'')!}{(\mu_i x+\mu_i' y+\mu_i'')!(\nu_i x-\nu_i' y+\nu_i'')!}z^y$$

Theorem. Assume T is proper hypergeom. . Then the order of the minimal telescoper for T w.r.t. y is no more than

$$B_{AZ} = \max\left\{\sum_{i=1}^{m}(\alpha_i'+\nu_i'), \sum_{i=1}^{m}(\beta_i'+\mu_i')\right\}.$$

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New}	B_{AZ}
non-proper		
example		

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper		
example		

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper	B_{New}	
example		

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\vert\vert$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper	B_{New}	?
example		

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper	B_{New}	?
T_1	9	10

Example.

$$T_1 = \frac{(x+3y)!(x-3y)!}{(5x+3y)(3x-y)(4x-3y)!(5x+3y)!}.$$

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper	B_{New}	?
T_2	5	6

Example.

$$T_2 = \frac{(x+3y-1)!^2}{(3y-2x-4)!(2x+3y-2)!}.$$

New upper bound v.s. Apagodu-Zeilberger

	New	Apagodu-Zeilberger
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$	B_{AZ}
non-proper	B_{New}	?
T_3	3	?

Example.

$$T_3 = \frac{x^4 + x^3 y + 2x^2 y^2 + 2x^2 y + xy^2 + 2y^3 + x^2 + y^2 - x - y}{(x^2 + y + 1)(x^2 + y)(x + 2y)} y!$$

Lower bound

Example (cont.). $T = y!/(x + 2y)$.

Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Initial reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Initial reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$c_0(x)\left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) + \cdots + c_\rho(x)\left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) = 0$$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Initial reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Initial reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

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- Property. $b_i = x + 2y$ or $x + 2y + 1$.

Lower bound

Example (cont.). $T = y!/(x + 2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

PFD & $b_0 = x + 2y$

$$\Downarrow$$

$$\exists i, s.t.\ b_i = b_0$$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

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$$\Downarrow$$

$$\exists i, s.t.\ b_i = b_0$$

$$\Downarrow$$ Prop. $b_i = x + 2y$ or $x + 2y + 1$

minimal $i = 2$

Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

PFD & $b_0 = x + 2y$

$\Downarrow$

$\exists i, s.t.\ b_i = b_0$

$\Downarrow$ Prop. $b_i = x + 2y$ or $x + 2y + 1$

minimal $i = 2$

Conclusion. Lower bound is 2.

New lower bound

Theorem. Assume T has initial reduction

$$T = \Delta_y(\cdots) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H,$$

with b_0 integer-linear w.r.t. y. Then the order of the minimal telescoper for T w.r.t. y is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \quad \min_{h \in \mathbb{Z}} \left\{\rho \in \mathbb{N} \setminus \{0\} : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\right\}$$

Abramov-Le lower bound (2005)

Theorem. Assume T has initial reduction

$$T = \Delta_y(\dots) + \frac{a_0'}{b_0} H'$$

where

$$\frac{u'}{v'} := \frac{\sigma_y(H')}{H'}, \quad \frac{c'}{d'} := \frac{\sigma_x(H')}{H'},$$

with u', v', c', d' and b_0 integer-linear w.r.t. y. Then the order of the minimal telescoper for T w.r.t. y is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$$

New lower bound v.s. Abramov-Le

	New	Abramov-Le
lower bound	$\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \ \min_h$ $\left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$
example		

Example.

New lower bound v.s. Abramov-Le

	New	Abramov-Le
lower bound	$\max_p \ \min_h$ $\left\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\right\}$	$\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$
T_1	4	3

Example.

$$T_1 = \frac{1}{(x+3y+1)(5x-4y+4)(5x-4y+14)!}.$$

New lower bound v.s. Abramov-Le

	New	Abramov-Le
lower bound	$\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$
T_2	7	3

Example.

$$T_2 = \frac{1}{(x+3y+1)(5x-7y)(5x-7y+14)!}.$$

New lower bound v.s. Abramov-Le

	New	Abramov-Le
lower bound	$\max_p \ \min_h$ $\left\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\right\}$	$\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$
T_3	12	5

Example.

$$T_3 = \frac{1}{(x+5y+1)(5x-12y)(5x-12y+24)!}.$$

Summary

Result.

- Order bounds for telescopers

Future work.

- Creative telescoping for q-hypergeometric terms