Reduction and Creative Telescoping For Hypergeometric Terms

Hui Huang

KLMM, AMSS,
Chinese Academy of Sciences

Institute for Algebra,
Johannes Kepler University

Joint work with S. Chen, M. Kauers and Z. Li

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Background

$f(x, y)$ a bivariate hypergeometric term.

Hypergeometric summation. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x, y).$$

Hypergeometric identities. Prove the identity

$$\sum_{y=-\infty}^{\infty} f(x, y) = h(x).$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x, y).$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x, y).$$

Method. Gosper 1978, Abramov & Petkovšek 2001

$$f(x, y) = g(x, y + 1) - g(x, y)$$

$$\Downarrow$$

$$\sum_{y=-\infty}^{\infty} f(x, y) = g(x, \infty) - g(x, -\infty)$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x,y).$$

Method. Gosper1978, Abramov & Petkovšek2001, Wilf & Zeilberger1990

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i f(x,y) = g(x,y+1) - g(x,y)$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x,y).$$

Method. Gosper1978, Abramov & Petkovšek2001, Wilf & Zeilberger1990

$$\underbrace{\sum_{i=0}^{\rho} e_i(x)\sigma_x^i}_{\text{telescoper}} f(x,y) = g(x,y+1) - \underbrace{g(x,y)}_{\text{certificate}}$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x, y).$$

Method. Gosper1978, Abramov & Petkovšek2001, Wilf & Zeilberger1990

$$\boxed{\sum_{i=0}^{\rho} e_i(x)\sigma_x^i} f(x, y) = g(x, y+1) - \boxed{g(x, y)}$$

telescoper

certificate

$$\Downarrow$$

$$\sum_{i=0}^{\rho} e_i(x) \left(\sum_{y=-\infty}^{\infty} f(x, y) \right) = g(x, \infty) - g(x, -\infty)$$

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x, y).$$

Method. Gosper1978, Abramov & Petkovšek2001, Wilf & Zeilberger1990

$$\boxed{\sum_{i=0}^{\rho} e_i(x)\sigma_x^i} f(x, y) = g(x, y+1) - \boxed{g(x, y)}$$

telescoper

certificate

$$\Downarrow$$

$$\sum_{i=0}^{\rho} e_i(x) \left(\sum_{y=-\infty}^{\infty} f(x, y) \right) = g(x, \infty) - g(x, -\infty)$$

$$+$$

Petkovšek 1992

Hypergeometric summation

Problem. Find the "closed form" of

$$\sum_{y=-\infty}^{\infty} f(x,y).$$

Method. Gosper1978, Abramov & Petkovšek2001, Wilf & Zeilberger1990

$$\underbrace{\sum_{i=0}^{\rho} e_i(x)\sigma_x^i}_{\text{telescoper}} f(x,y) = g(x,y+1) - \underbrace{g(x,y)}_{\text{certificate}}$$

$$\Downarrow$$

$$\sum_{i=0}^{\rho} e_i(x)\left(\sum_{y=-\infty}^{\infty} f(x,y)\right) = \underbrace{g(x,\infty) - g(x,-\infty)}_{\text{Usually it is zero}}$$

$$+$$

Petkovšek 1992

Hypergeometric identities

Problem. Prove the identity

$$\sum_{y=-\infty}^{\infty} f(x, y) = h(x).$$

Method. Wilf & Zeilberger1990

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i \left(\sum_{y=-\infty}^{\infty} f(x, y)\right) = g(x, \infty) - g(x, -\infty)$$

Hypergeometric identities

Problem. Prove the identity

$$\sum_{y=-\infty}^{\infty} f(x,y) = h(x).$$

Method. Wilf & Zeilberger1990

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i \left(\sum_{y=-\infty}^{\infty} f(x,y)\right) = g(x,\infty) - g(x,-\infty)$$

$$+$$

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i\, h(x) = g(x,\infty) - g(x,-\infty) \text{ \& initial values match}$$

Hypergeometric identities

Problem. Prove the identity

$$\sum_{y=-\infty}^{\infty} f(x,y) = h(x).$$

Method. Wilf & Zeilberger1990

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i \left(\sum_{y=-\infty}^{\infty} f(x,y)\right) = g(x,\infty) - g(x,-\infty)$$

$$+$$

$$\sum_{i=0}^{\rho} e_i(x)\sigma_x^i \, h(x) = g(x,\infty) - g(x,-\infty) \text{ \& initial values match}$$

Key. Compute a telescoper!

Generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)

2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)

3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)

Motivating example

Consider

$$T = \frac{y^{10}}{x + y}$$

- The minimal telescoper for T w.r.t. y is

$$L = \sigma_x - \frac{1}{x^{10}}(x + 1)^{10}$$

Certificate for the example

$$
\begin{aligned}
G = \frac{1}{10} \Bigg(& -1/21 \, \frac{x^3 \left(175 x^7 + 700 x^6 + 1234 x^5 + 1252 x^4 + 790 x^3 + 310 x^2 + 70 x + 7\right)}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{42} \, \frac{x \left(1750 x^7 + 5950 x^6 + 9558 x^5 + 9186 x^4 + 5630 x^3 + 2180 x^2 + 490 x + 49\right) y^2}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{18} \, \frac{\left(990 x^9 + 3960 x^8 + 7890 x^7 + 10260 x^6 + 9654 x^5 + 6780 x^4 + 3490 x^3 + 1240 x^2 + 270 x + 27\right) y^3}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{36} \, \frac{x \left(792 x^7 + 2574 x^6 + 4020 x^5 + 3801 x^4 + 2310 x^3 + 891 x^2 + 200 x + 20\right) y^4}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{1}{12} \, \frac{\left(1320 x^9 + 5280 x^8 + 11352 x^7 + 16566 x^6 + 17540 x^5 + 13535 x^4 + 7410 x^3 + 2721 x^2 + 600 x + 60\right) y^5}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{6} \, \frac{x \left(660 x^7 + 1980 x^6 + 2948 x^5 + 2717 x^4 + 1630 x^3 + 625 x^2 + 140 x + 14\right) y^6}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{42} \, \frac{\left(4620 x^9 + 18480 x^8 + 42900 x^7 + 68640 x^6 + 78188 x^5 + 63305 x^4 + 35630 x^3 + 13265 x^2 + 2940 x + 294\right) y^7}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{84} \, \frac{x \left(924 x^7 + 2310 x^6 + 3168 x^5 + 2805 x^4 + 1650 x^3 + 627 x^2 + 140 x + 14\right) y^8}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{36} \, \frac{\left(660 x^9 + 2640 x^8 + 6732 x^7 + 11550 x^6 + 13728 x^5 + 11385 x^4 + 6490 x^3 + 2431 x^2 + 540 x + 54\right) y^9}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{9} \, \frac{\left(495 x^9 + 2145 x^8 + 5610 x^7 + 9702 x^6 + 11550 x^5 + 9570 x^4 + 5445 x^3 + 2035 x^2 + 451 x + 45\right) y^{10}}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} + y^{11} \Bigg) \\
& \cdot \left(10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1\right) (x + y)^{-1}
\end{aligned}
$$

Certificate for the example

$$
\begin{aligned}
G = \frac{1}{10} \Bigg(& -1/21 \, \frac{x^3 (175 x^7 + 700 x^6 + 1234 x^5 + 1252 x^4 + 790 x^3 + 310 x^2 + 70 x + 7)}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{42} \frac{x (1750 x^7 + 5950 x^6 + 9558 x^5 + 9186 x^4 + 5630 x^3 + 2180 x^2 + 490 x + 49) y^2}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{18} \frac{(990 x^9 + 3960 x^8 + 7890 x^7 + 10260 x^6 + 9654 x^5 + 6780 x^4 + 3490 x^3 + 1240 x^2 + 270 x + 27) y^3}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{36} \frac{x (792 x^7 + 2574 x^6 + 4020 x^5 + 3801 x^4 + 2310 x^3 + 891 x^2 + 200 x + 20) y^4}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{1}{12} \frac{(1320 x^9 + 5280 x^8 + 11352 x^7 + 16566 x^6 + 17540 x^5 + 13535 x^4 + 7410 x^3 + 2721 x^2 + 600 x + 60) y^5}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{6} \frac{x (660 x^7 + 1980 x^6 + 2948 x^5 + 2717 x^4 + 1630 x^3 + 625 x^2 + 140 x + 14) y^6}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{42} \frac{(4620 x^9 + 18480 x^8 + 42900 x^7 + 68640 x^6 + 78188 x^5 + 63305 x^4 + 35630 x^3 + 132\cdots x^2 + 2940 x + 294) y^7}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{84} \frac{x (924 x^7 + 2310 x^6 + 3168 x^5 + 2805 x^4 + 1650 x^3 + 627 x^2 + 140 x + 14) y^8}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& + \frac{5}{36} \frac{(660 x^9 + 2640 x^8 + 6732 x^7 + 11550 x^6 + 13728 x^5 + 11385 x^4 + 6490 x^3 + 2431 x^2 + 540 x + 54) y^9}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} \\
& - \frac{1}{9} \frac{(495 x^9 + 2145 x^8 + 5610 x^7 + 9702 x^6 + 11550 x^5 + 9570 x^4 + 5445 x^3 + 2035 x^2 + 451 x + 45) y^{10}}{10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1} + y^{11} \Bigg) \\
& \cdot \left(10 x^9 + 45 x^8 + 120 x^7 + 210 x^6 + 252 x^5 + 210 x^4 + 120 x^3 + 45 x^2 + 10 x + 1\right) (x + y)^{-1}
\end{aligned}
$$

Very often, certificates are not needed!

The fourth generation

Reduction-based creative telescoping (since $\approx$ 2010)

The fourth generation

Reduction-based creative telescoping (since $\approx$ 2010)

- Differential case:
  - Bostan, Chen, Chyzak, Li (2010): bivariate rational functions
  - Chen, Kauers, Singer (2012): triple rational functions
  - Bostan, Lairez, Salvy (2013): multivariate rational function
  - Bostan, Chen, Chyzak, Li, Xin (2013): bivariate hyperexp. fun.
  - Chen, Kauers, Koutschan (2016): bivariate algebraic functions

The fourth generation

Reduction-based creative telescoping (since ≈ 2010)

- Differential case:
  - Bostan, Chen, Chyzak, Li (2010): bivariate rational functions
  - Chen, Kauers, Singer (2012): triple rational functions
  - Bostan, Lairez, Salvy (2013): multivariate rational function
  - Bostan, Chen, Chyzak, Li, Xin (2013): bivariate hyperexp. fun.
  - Chen, Kauers, Koutschan (2016): bivariate algebraic functions
- Shift case: ???

The fourth generation

Reduction-based creative telescoping (since ≈ 2010)

- Differential case:
  - Bostan, Chen, Chyzak, Li (2010): bivariate rational functions
  - Chen, Kauers, Singer (2012): triple rational functions
  - Bostan, Lairez, Salvy (2013): multivariate rational function
  - Bostan, Chen, Chyzak, Li, Xin (2013): bivariate hyperexp. fun.
  - Chen, Kauers, Koutschan (2016): bivariate algebraic functions
- Shift case:
  - Chen, Huang, Kauers, Li (2015): bivariate hypergeom. terms
  - Huang (2016): new bounds for hypergeom. creative telescoping

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Hypergeometric summability

$\mathbb{C}$ the field of complex numbers.

Definition. A nonzero term $T(y)$ is hypergeometric over $\mathbb{C}(y)$ if $T(y+1)/T(y) \in \mathbb{C}(y)$.

Examples. $f(y) \in \mathbb{C}(y) \setminus \{0\}$, c^y with $c \in \mathbb{C} \setminus \{0\}$, $y!$, etc.

Hypergeometric summability

$\mathbb{C}$ the field of complex numbers.

Definition. A nonzero term $T(y)$ is hypergeometric over $\mathbb{C}(y)$ if $T(y+1)/T(y) \in \mathbb{C}(y)$.

Examples. $f(y) \in \mathbb{C}(y) \setminus \{0\}$, c^y with $c \in \mathbb{C} \setminus \{0\}$, $y!$, etc.

Definition. A hypergeom. term $T(y)$ is summable if

$$T(y) = G(y+1) - G(y) \text{ for some hypergeom. term } G(y).$$

Example. $y \cdot y! = (y+1)! - y!$ is summable; but $y!$ is not.

Multiplicative decomposition

Notation.

- $\sigma_y(T(y)) = T(y+1)$, $\Delta_y(T) = \sigma_y(T) - T$ for a term $T(y)$.
- f_d and f_n: the denominator and numerator of $f \in \mathbb{C}(y)$.

Multiplicative decomposition

Notation.

- $\sigma_y(T(y)) = T(y+1)$, $\Delta_y(T) = \sigma_y(T) - T$ for a term $T(y)$.
- f_d and f_n: the denominator and numerator of $f \in \mathbb{C}(y)$.

For a hypergeom. term $T(y)$, $\exists\ S \in \mathbb{C}(y)$ and H hypergeom. s.t.

- $T(y) = S(y) \cdot H(y)$
- $K := \sigma_y(H)/H$ is shift-reduced, i.e.

$$\gcd\left(K_d, \sigma_y^\ell(K_n)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z}.$$

Call K a kernel of T, and S the corr. shell.

Multiplicative decomposition

Notation.

- $\sigma_y(T(y)) = T(y+1)$, $\Delta_y(T) = \sigma_y(T) - T$ for a term $T(y)$.
- f_d and f_n: the denominator and numerator of $f \in \mathbb{C}(y)$.

For a hypergeom. term $T(y)$, $\exists\, S \in \mathbb{C}(y)$ and H hypergeom. s.t.

- $T(y) = S(y) \cdot H(y)$ — multi. decomposition
- $K := \sigma_y(H)/H$ is shift-reduced, i.e.

$$\gcd\left(K_d, \sigma_y^{\ell}(K_n)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z}.$$

Call K a kernel of T, and S the corr. shell.

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Abramov-Petkovšek reduction (2001)

Let $T(y)$ be hypergeom. with a kernel K and shell S. Then

$$T = \underbrace{\Delta_y\left(\cdots\right)}_{\text{summable}} + \underbrace{\left(\frac{a}{b} + \frac{p}{K_d}\right) H}_{\text{possibly summable}},$$

where $H = T/S$, and $a, b, p \in \mathbb{C}[y]$ satisfy proper, shift-free, and strongly-prime conditions.

Abramov-Petkovšek reduction (2001)

Let $T(y)$ be hypergeom. with a kernel K and shell S. Then

$$T = \underbrace{\Delta_y\left(\cdots\right)}_{\text{summable}} + \underbrace{\left(\frac{a}{b} + \frac{p}{K_d}\right)H}_{\text{possibly summable}},$$

where $H = T/S$, and $a, b, p \in \mathbb{C}[y]$ satisfy proper, shift-free, and strongly-prime conditions.

Proposition. T is summable iff

- $a = 0$,
- $K_n z(y+1) - K_d z(y) = p$ has a solution in $\mathbb{C}[y]$.

Question

Can one determine hypergeometric summability directly without solving any equations?

Question

Can one determine hypergeometric summability directly without solving any equations?

Known results:

- Hyperexponentional Hermite reduction (Bostan et. al 2013)
- Rational Abramov reduction (1995)

Polynomial reduction

Let $K \in \mathbb{C}(y)$ be shift-reduced, define

- polynomial reduction map (w.r.t. K):

$$\begin{array}{rccc} \phi_K : & \mathbb{C}[y] & \longrightarrow & \mathbb{C}[y] \\ & p & \longmapsto & K_n \sigma_y(p) - K_d p. \end{array}$$

- standard complement of $\operatorname{im}(\phi_K)$:

$$\mathbb{W}_K = \operatorname{span}_{\mathbb{C}} \left\{ y^i \,|\, i \neq \deg_y(p) \text{ for all } p \in \operatorname{im}(\phi_K) \right\}.$$

Polynomial reduction

Let $K \in \mathbb{C}(y)$ be shift-reduced, define

- polynomial reduction map (w.r.t. K):

$$\begin{array}{rccc} \phi_K : & \mathbb{C}[y] & \longrightarrow & \mathbb{C}[y] \\ & p & \longmapsto & K_n \sigma_y(p) - K_d p. \end{array}$$

- standard complement of $\mathrm{im}(\phi_K)$:

$$\mathbb{W}_K = \mathrm{span}_{\mathbb{C}} \left\{ y^i \,|\, i \neq \deg_y(p) \text{ for all } p \in \mathrm{im}\,(\phi_K) \right\}.$$

Proposition. $\mathbb{C}[y] = \mathrm{im}(\phi_K) \oplus \mathbb{W}_K$.

Modified Abramov-Petkovšek reduction (2015)

- Abramov-Petkovšek reduction:

$$T = \Delta_y\left(\cdots\right) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H$$

Modified Abramov-Petkovšek reduction (2015)

- Abramov-Petkovšek reduction:

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H$$

- Polynomial reduction:

$$p \;\begin{cases} \nearrow\; p_1 \;\in \mathrm{im}(\phi_K) \\ \quad\;\; + \\ \searrow\; p_2 \;\in \mathbb{W}_K \end{cases}$$

Modified Abramov-Petkovšek reduction (2015)

- Abramov-Petkovšek reduction:

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H$$

- Polynomial reduction:

$$p \;\nearrow\; p_1 \in \mathrm{im}(\phi_K) \quad + \quad \searrow\; p_2 \in \mathbb{W}_K$$

$$\Downarrow$$

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a}{b} + \frac{p_2}{K_d}\right) H$$

Modified Abramov-Petkovšek reduction (2015)

- Abramov-Petkovšek reduction:

$$T = \Delta_y\left(\cdots\right) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H$$

- Polynomial reduction:

$$p \to p_1 \in \operatorname{im}(\phi_K)$$
$$+$$
$$p \to p_2 \in \mathbb{W}_K$$
$$\Downarrow$$
$$T = \Delta_y\left(\cdots\right) + \left(\frac{a}{b} + \frac{p_2}{K_d}\right) H$$

Proposition.

- T is summable iff $a = p_2 = 0$.
- # nonzero terms of $p_2 \leq \max\left(\deg_y(K_n), \deg_y(K_d)\right) - [\![\cdots]\!]$.

Iverson bracket

Modified Abramov-Petkovšek reduction (2015)

- Abramov-Petkovšek reduction:

$$T = \Delta_y\left(\cdots\right) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H$$

- Polynomial reduction:

$$p \nearrow p_1 \in \mathrm{im}(\phi_K) \quad + \quad \searrow p_2 \in \mathbb{W}_K$$

$$\Downarrow$$

$$T = \Delta_y\left(\cdots\right) + \underbrace{\left(\frac{a}{b} + \frac{p_2}{K_d}\right)}_{\text{a residual form (w.r.t. K)}} H$$

Proposition.

- T is summable iff $a = p_2 = 0$.
- # nonzero terms of $p_2 \leq \max\left(\deg_y(K_n), \deg_y(K_d)\right) - [\![\cdots]\!]$.

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

A.-P. $\searrow$

$$\boxed{T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p} \cdot H}$$

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

A.-P. $\searrow$

$$\boxed{T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p} \cdot H}$$

$\swarrow$

$$\boxed{\begin{array}{c} K_n z(y+1) - K_d z(y) = p \\ \Downarrow \\ z(y) \notin \mathbb{C}[y] \end{array}}$$

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

A.-P. $\searrow$

$$T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p} \cdot H$$

$\swarrow$

$$K_n z(y+1) - K_d z(y) = p$$

$$\Downarrow$$

$$z(y) \notin \mathbb{C}[y]$$

$\searrow$

$(y^3 + 1) \cdot y!$ is not summable

Example

Consider $T = (y^3+1)\cdot y!$, $K = y+2$ and $H = (y+1)y!$.

A.-P. ↘ ↙ Modified A.-P.

$$T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p}\cdot H$$

⇙

$$K_n z(y+1) - K_d z(y) = p$$

$$\Downarrow$$

$$z(y) \notin \mathbb{C}[y]$$

⇘

$(y^3+1)\cdot y!$ is not summable

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

A.-P. ↘ ↙ Modified A.-P.

$$T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p} \cdot H$$

⇙ (A.-P.)

$$K_n z(y+1) - K_d z(y) = p$$
$$\Downarrow$$
$$z(y) \notin \mathbb{C}[y]$$

⇘ (Modified A.-P.)

$$p = 2 \mod \operatorname{im}(\phi_K)$$
$$\Downarrow$$
$$T = \Delta_y(\cdots) + 2 \cdot H$$

⇘

$(y^3 + 1) \cdot y!$ is not summable

Example

Consider $T = (y^3 + 1) \cdot y!$, $K = y + 2$ and $H = (y + 1)y!$.

A.-P. ↘ Modified A.-P. ↙

$$T = \Delta_y(\cdots) + \underbrace{(y^2 - y + 1)}_{p} \cdot H$$

⇙ ⇘

$$K_n z(y+1) - K_d z(y) = p$$
$$\Downarrow$$
$$z(y) \notin \mathbb{C}[y]$$

$$p = 2 \mod \operatorname{im}(\phi_K)$$
$$\Downarrow$$
$$T = \Delta_y(\cdots) + 2 \cdot H$$

⇘ ⇙

$(y^3 + 1) \cdot y!$ is not summable

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y\big(0\big) + \binom{x}{y}$.

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y\left(0\right) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y\left(0\right) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.
  - $\sum_{y=0}^{\infty} \binom{x}{y} = \frac{1}{2} + \sum_{y=0}^{\infty} \frac{x+1}{2(y+1)}\binom{x}{y} = \frac{1}{2}\sum_{y=0}^{\infty}\binom{x+1}{y}$.

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y(0) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.

$F(x)$ $\qquad$ $F(x+1)$

- $\sum_{y=0}^{\infty}\binom{x}{y} = \frac{1}{2} + \sum_{y=0}^{\infty}\frac{x+1}{2(y+1)}\binom{x}{y} = \frac{1}{2}\sum_{y=0}^{\infty}\binom{x+1}{y}.$

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y(0) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.
  - $F(x+1) - 2F(x) = 0$.

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y\left(0\right) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.
  - $F(x+1) - 2F(x) = 0$.
  - 2^x is a solution, and $2^0 = 1 = F(0)$.

A.-P. reduction vs modified A.-P. reduction

Example. Prove

$$\sum_{y=0}^{\infty} \binom{x}{y} = 2^x.$$

- A.-P. reduction: $\binom{x}{y} = \Delta_y\big(0\big) + \binom{x}{y}$.
- Modified reduction: $\binom{x}{y} = \Delta_y\left(-\frac{1}{2}\binom{x}{y}\right) + \frac{x+1}{2(y+1)}\binom{x}{y}$.
  - $F(x+1) - 2F(x) = 0$.
  - 2^x is a solution, and $2^0 = 1 = F(0)$.
  - $F(x) = 2^x$.

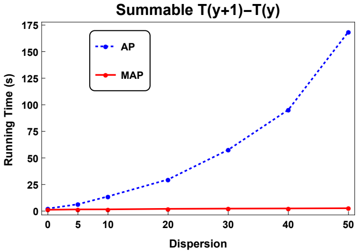
Timings (in seconds)

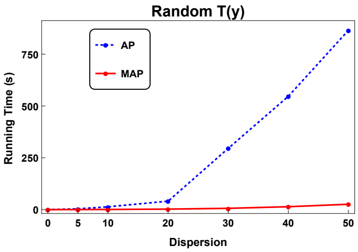
Consider

$$T = \frac{f(y)}{g_1(y)g_2(y)} \prod_{k=n_0}^{y} \frac{u(k)}{v(k)},$$

where

- $g_i = p_i \sigma_y^{\lambda}(p_i) \sigma_y^{\mu}(p_i)$, $\lambda, \mu \in \mathbb{N}$ and $\lambda \leq \mu$, $n_0 \in \mathbb{F}$
- $f, p_i \in \mathbb{Z}[y]$, $\deg(p_1) = \deg(p_2) = 10$ and $\deg(f) = 20$.





Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Bivariate hypergeometric terms

Definition. A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if $\sigma_x(T)/T,\ \sigma_y(T)/T\ \in \mathbb{C}(x, y)$.

Telescoping problem. Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in \mathbb{C}(x)\langle\sigma_x\rangle$ s.t.

$$L \cdot T = \Delta_y(G) \quad \text{for some hypergeom. term} \quad G(x, y).$$

Bivariate hypergeometric terms

Definition. A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if $\sigma_x(T)/T,\ \sigma_y(T)/T\ \in \mathbb{C}(x, y)$.

Telescoping problem. Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in \mathbb{C}(x)\langle\sigma_x\rangle$ s.t.

$$L \cdot T = \Delta_y(G) \quad \text{for some hypergeom. term } G(x, y).$$

telescoper (L) — certificate (G)

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$T = \Delta_y\Big(\cdots\Big) + r_0 H$$

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$T = \Delta_y\Big(\cdots\Big) + r_0 H$$

$$\sigma_x(T) = \Delta_y\Big(\cdots\Big) + r_1 H$$

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$
\begin{aligned}
T &= \Delta_y\Big(\cdots\Big) + r_0 H \\
\sigma_x(T) &= \Delta_y\Big(\cdots\Big) + r_1 H \\
\sigma_x^2(T) &= \Delta_y\Big(\cdots\Big) + r_2 H \\
&\vdots \\
\sigma_x^\rho(T) &= \Delta_y\Big(\cdots\Big) + r_\rho H
\end{aligned}
$$

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, r_0 H$$

$$c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, r_1 H$$

$$c_2(x)\, \sigma_x^2(T) = \Delta_y\Big(\cdots\Big) + c_2(x)\, r_2 H$$

$$\vdots$$

$$c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, r_\rho H$$

Telescoping via reduction

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$
+\begin{cases}
c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, r_0 H \\
c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, r_1 H \\
c_2(x)\, \sigma_x^2(T) = \Delta_y\Big(\cdots\Big) + c_2(x)\, r_2 H \\
\vdots \\
c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, r_\rho H
\end{cases}
$$

$$
\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T) = \Delta_y\Big(\cdots\Big) + \left(\sum_{j=0}^{\rho} c_j(x) r_j\right) H
$$

Problem

$$\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T) = \Delta_y\left(\cdots\right) + \left(\sum_{j=0}^{\rho} c_j(x) r_j\right) H$$

Problem

$$\underbrace{\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T)}_{\text{a telescoper for } T} = \Delta_y\left(\cdots\right) + \underbrace{\left(\sum_{j=0}^{\rho} c_j(x) r_j\right)}_{=0} H$$

$$\overset{?}{\Longleftrightarrow}$$

Problem

$$\underbrace{\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T)}_{\text{a telescoper for } T} = \Delta_y\left(\cdots\right) + \underbrace{\left(\sum_{j=0}^{\rho} c_j(x) r_j\right)}_{=0} H$$

$$\Longleftarrow$$

Problem

$$\underbrace{\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T)}_{\text{a telescoper for } T} = \Delta_y\left(\cdots\right) + \underbrace{\left(\sum_{j=0}^{\rho} c_j(x) r_j\right)}_{=0} H$$

$$\Longleftarrow$$

$$\not\Longrightarrow$$

Problem

$$\underbrace{\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T)}_{\text{a telescoper for } T} = \Delta_y\left(\cdots\right) + \underbrace{\left(\sum_{j=0}^{\rho} c_j(x) r_j\right)}_{=0} H$$

$\Longleftarrow$

$\not\Longrightarrow$

$\sum_{j=0}^{\rho} c_j r_j$
may not be a residual form

Sum of residual forms

Example. Let H be hypergeom. with $K = \sigma_y(H)/H = 1/y$.

$$\frac{1}{2y+1} + \frac{1}{2y+3} = \frac{4(1+y)}{(2y+1)(2y+3)}$$ is not a residual form.

Sum of residual forms

Example. Let H be hypergeom. with $K = \sigma_y(H)/H = 1/y$.

$$\frac{1}{2y+1} + \frac{1}{2y+3} = \frac{4(1+y)}{(2y+1)(2y+3)} \text{ is not a residual form.}$$

$$\begin{aligned} &\frac{1}{2y+1}H + \frac{1}{2y+3}H \\ = &\frac{1}{2y+1}H + \left(\Delta_y\left(\cdots\right) + \left(-\frac{3}{2(2y+1)} + \frac{1}{2y}\right)H\right) \\ = &\Delta_y\left(\cdots\right) + \underbrace{\left(-\frac{1}{2(2y+1)} + \frac{1}{2y}\right)}_{\text{a residual form w.r.t. } K} H \end{aligned}$$

Sum of residual forms

Example. Let H be hypergeom. with $K = \sigma_y(H)/H = 1/y$.

$$\frac{1}{2y+1} + \frac{1}{2y+3} = \frac{4(1+y)}{(2y+1)(2y+3)} \text{ is not a residual form.}$$

$$\begin{aligned} & \frac{1}{2y+1}H + \frac{1}{2y+3}H \\ = & \frac{1}{2y+1}H + \left(\Delta_y(\cdots) + \left(-\frac{3}{2(2y+1)} + \frac{1}{2y}\right)H\right) \\ = & \Delta_y(\cdots) + \underbrace{\left(-\frac{1}{2(2y+1)} + \frac{1}{2y}\right)}_{\text{a residual form w.r.t. } K} H \end{aligned}$$

Theorem. r, s residual forms w.r.t. K, $\exists$ a residual form t s.t.

$$sH = \Delta_y\Big(\dots\Big) + tH \quad \text{and} \quad r + t \quad \text{is a residual form.}$$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, r_0 H$$

$$c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, r_1 H$$

$$c_2(x)\, \sigma_x^2(T) = \Delta_y\Big(\cdots\Big) + c_2(x)\, r_2 H$$

$$\vdots$$

$$c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, r_\rho H$$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, r_0' H$$

$$c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, r_1' H$$

$$c_2(x)\, \sigma_x^2(T) = \Delta_y\Big(\cdots\Big) + c_2(x)\, r_2' H$$

$$\vdots$$

$$c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, r_\rho' H$$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$
+\begin{cases}
c_0(x)\, T = \Delta_y\left(\cdots\right) + c_0(x)\, r_0' H \\
c_1(x)\, \sigma_x(T) = \Delta_y\left(\cdots\right) + c_1(x)\, r_1' H \\
c_2(x)\, \sigma_x^2(T) = \Delta_y\left(\cdots\right) + c_2(x)\, r_2' H \\
\vdots \\
c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\left(\cdots\right) + c_\rho(x)\, r_\rho' H
\end{cases}
$$

$$
\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T) = \Delta_y\left(\cdots\right) + \left(\sum_{j=0}^{\rho} c_j(x) r_j'\right) H
$$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$+\left\{\begin{aligned} c_0(x)\,T &= \Delta_y\Big(\cdots\Big) + c_0(x)\,r_0'H \\ c_1(x)\,\sigma_x(T) &= \Delta_y\Big(\cdots\Big) + c_1(x)\,r_1'H \\ c_2(x)\,\sigma_x^2(T) &= \Delta_y\Big(\cdots\Big) + c_2(x)\,r_2'H \\ &\vdots \\ c_\rho(x)\,\sigma_x^\rho(T) &= \Delta_y\Big(\cdots\Big) + c_\rho(x)\,r_\rho'H \end{aligned}\right.$$

$$\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T) = \Delta_y\Big(\cdots\Big) + \left(\sum_{j=0}^{\rho} c_j(x) r_j'\right) H$$

telescoper?

$\stackrel{?}{=} 0$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x) r_0' + c_1(x) r_1' + \cdots + c_\rho(x) r_\rho' \stackrel{?}{=} 0$$

$$\Downarrow$$

a linear system with unknowns $c_j(x)$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x)r_0' + c_1(x)r_1' + \cdots + c_\rho(x)r_\rho' \stackrel{?}{=} 0$$

$$\Downarrow$$

a linear system with unknowns $c_j(x)$

$$\Downarrow$$

a telescoper $c_0(x) + c_1(x)\sigma_x + \cdots + c_\rho(x)\sigma_x^\rho$

Telescoping via reduction (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T(x, y)$ of order ρ w.r.t. y.

Idea. Let K and S be a kernel and shell of T, and $H = T/S$.

$$c_0(x)r_0' + c_1(x)r_1' + \cdots + c_\rho(x)r_\rho' \stackrel{?}{=} 0$$

$$\Downarrow$$

a linear system with unknowns $c_j(x)$

$$\Downarrow$$

a telescoper $c_0(x) + c_1(x)\sigma_x + \cdots + c_\rho(x)\sigma_x^\rho$

Remarks.

- The first linear depend. leads to a minimal telescoper.
- One can leave the certificate as an un-normalized sum.

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1 Compute a kernel K and shell S. Set $H = T/S$.

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1 Compute a kernel K and shell S. Set $H = T/S$.

2 Apply modified A.-P. reduction w.r.t. y: $T = \Delta_y(\cdots) + rH$.

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1 Compute a kernel K and shell S. Set $H = T/S$.

2 Apply modified A.-P. reduction w.r.t. y: $T = \Delta_y(\cdots) + rH$.

3 If $r = 0$, return $L = 1$.

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1 Compute a kernel K and shell S. Set $H = T/S$.

2 Apply modified A.-P. reduction w.r.t. y: $T = \Delta_y(\cdots) + rH$.

3 If $r = 0$, return $L = 1$.

4 If r_d is not integer-linear, return " No telescoper exists!" .

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1. Compute a kernel K and shell S. Set $H = T/S$.
2. Apply modified A.-P. reduction w.r.t. y: $T = \Delta_y(\cdots) + rH$.
3. If $r = 0$, return $L = 1$.
4. If r_d is not integer-linear, return " No telescoper exists!" .

Existence criterion

Outline of reduction-based telescoping

Algorithm. Given a hypergeom. term $T(x, y)$, compute a minimal telescoper L for T w.r.t. y.

1 Compute a kernel K and shell S. Set $H = T/S$.

2 Apply modified A.-P. reduction w.r.t. y: $T = \Delta_y(\cdots) + rH$.

3 If $r = 0$, return $L = 1$.

4 If r_d is not integer-linear, return " No telescoper exists!" .

5 For $\rho = 1, 2, \ldots$ do

find a telescoper L for T of order ρ and return L.

Example

Consider

$$T = \frac{1}{x + 2y} \cdot y!$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

- A kernel $K = y + 1$ and shell $S = 1/(x + 2y)$;
- $H = T/S = y!$.

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

$$\sigma_x(T) = \Delta_y(g_1) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\left(\frac{2}{x+2y} + \frac{0}{K_d}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x) \cdot \left(\frac{2}{x+2y} + \frac{0}{K_d} \right)$$

$$+ c_1(x) \cdot \left(\frac{2}{x+2y+1} + \frac{0}{K_d} \right)$$

$$= 0$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x)\cdot \left(\frac{2}{x+2y} + \frac{0}{K_d} \right)$$

$$+ c_1(x)\cdot \left(\frac{2}{x+2y+1} + \frac{0}{K_d} \right)$$

No solution in $\mathbb{C}(x)$!

$$= 0$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)H$$

$$\sigma_x^2(T) = \Delta_y\left(g_2\right) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)H$$

$$\sigma_x^2(T) = \Delta_y\left(g_2\right) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d}\right)H$$

$$\sigma_x^3(T) = \Delta_y\left(g_3\right) + \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{K_d}\right)H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{K_d}\right) H$$

$$\sigma_x(T) = \Delta_y(g_1) + \left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right) H$$

$$\sigma_x^2(T) = \Delta_y(g_2) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d}\right) H$$

$$\sigma_x^3(T) = \Delta_y(g_3) + \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{K_d}\right) H$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\left(\frac{2}{x+2y} + \frac{0}{K_d}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{K_d}\right)$$

$$\left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d}\right)$$

$$\left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{K_d}\right)$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\begin{aligned} & c_0(x) \cdot \left(\frac{2}{x+2y} + \frac{0}{K_d} \right) \\ & + c_1(x) \cdot \left(\frac{2}{x+2y+1} + \frac{0}{K_d} \right) \\ & + c_2(x) \cdot \left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d} \right) \\ & + c_3(x) \cdot \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{K_d} \right) \\ & = 0 \end{aligned}$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$
\begin{aligned}
& - 2 \cdot \left(\frac{2}{x+2y} + \frac{0}{K_d} \right) \\
& + 2 \cdot \left(\frac{2}{x+2y+1} + \frac{0}{K_d} \right) \\
& - x \cdot \left(-\frac{-4/x}{x+2y} + \frac{2/x}{K_d} \right) \\
& + (x+1) \cdot \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{K_d} \right) \\
& = 0
\end{aligned}
$$

Example

Consider

$$T = \frac{1}{x + 2y} \cdot y!$$

Therefore,

- the minimal telescoper for T w.r.t. y is

$$L = (x+1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

Therefore,

- the minimal telescoper for T w.r.t. y is

$$L = (x+1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

- the corresponding certificate is

$$\begin{aligned} G &= (x+1) \cdot g_3 - x \cdot g_2 + 2 \cdot g_1 - 2 \cdot g_0 \\ &= \frac{2y!}{(x+2y)(x+2y+1)} \end{aligned}$$

Timing (in seconds)

Consider

$$T = \frac{f(x,y)}{g_1(x+y)g_2(2x+y)} \frac{\Gamma(2\alpha x + y)}{\Gamma(x+\alpha y)}$$

where

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu), \ \alpha, \lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

$(m, n, \alpha, \lambda, \mu)$	Zeilberger	RCT+cert	RCT	order
$(2, 0, 1, 5, 10)$	354.46	58.01	4.93	4
$(2, 0, 2, 5, 10)$	576.31	363.25	53.15	6
$(2, 0, 3, 5, 10)$	2989.18	1076.50	197.75	7
$(2, 3, 3, 5, 10)$	3074.08	1119.26	223.41	7
$(3, 0, 1, 5, 10)$	18946.80	407.06	43.01	6
$(3, 0, 2, 5, 10)$	46681.30	2040.21	465.88	8
$(3, 0, 3, 5, 10)$	172939.00	5970.10	1949.71	9

Timing (in seconds)

Consider

$$T = \frac{f(x,y)}{g_1(x+y)g_2(2x+y)} \frac{\Gamma(2\alpha x + y)}{\Gamma(x + \alpha y)}$$

where

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\alpha, \lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

$(m, n, \alpha, \lambda, \mu)$	Zeilberger	RCT+cert	RCT	order
$(2, 0, 1, 5, 10)$	354.46	58.01	4.93	4
$(2, 0, 2, 5, 10)$	576.31	363.25	53.15	6
$(2, 0, 3, 5, 10)$	2989.18	1076.50	197.75	7
$(2, 3, 3, 5, 10)$	3074.08	1119.26	223.41	7
$(3, 0, 1, 5, 10)$	18946.80	407.06	43.01	6
$(3, 0, 2, 5, 10)$	46681.30	2040.21	465.88	8
$(3, 0, 3, 5, 10)$	172939.00	5970.10	1949.71	9

Timing (in seconds)

Consider

$$T = \frac{f(x,y)}{g_1(x+y)g_2(2x+y)} \frac{\Gamma(2\alpha x + y)}{\Gamma(x+\alpha y)}$$

where

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\alpha, \lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

$(m, n, \alpha, \lambda, \mu)$	Zeilberger	RCT+cert	RCT	order
$(2, 0, 1, 5, 10)$	354.46	58.01	4.93	4
$(2, 0, 2, 5, 10)$	576.31	363.25	53.15	6
$(2, 0, 3, 5, 10)$	2989.18	1076.50	197.75	7
$(2, 3, 3, 5, 10)$	3074.08	1119.26	223.41	7
$(3, 0, 1, 5, 10)$	18946.80	407.06	43.01	6
$(3, 0, 2, 5, 10)$	46681.30	2040.21	465.88	8
$(3, 0, 3, 5, 10)$	172939.00	5970.10	1949.71	9

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

New upper bound

Theorem. Assume T has initial reduction

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{K_d}\right) H$$

with $b_0 = c\prod_{i=1}^{m}\prod_{k=0}^{d_i}(\alpha_i x + \beta_i y + \gamma_i + k)^{c_{ik}}$, and for each $i \neq j$, either

$$\alpha_i \neq \alpha_j \quad \text{or} \quad \beta_i \neq \beta_j \quad \text{or} \quad \gamma_i - \gamma_j \notin \mathbb{Z}.$$

Then the order of a minimal telescoper for T w.r.t. y is no more than

$$\begin{aligned} B_{New} := & \max\{\deg_y(K_n), \deg_y(K_d)\} \\ & - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!] \\ & + \sum_{i=1}^{m} \beta_i \cdot \max_{0 \leq k \leq d_i}\{c_{ik}\}. \end{aligned}$$

Apagodu-Zeilberger upper bound (2005)

Definition. A hypergeom. term T is said to be proper if

$$T = p(x, y) \prod_{i=1}^{m} \frac{(\alpha_i x + \alpha_i' y + \alpha_i'')!(\beta_i x - \beta_i' y + \beta_i'')!}{(\mu_i x + \mu_i' y + \mu_i'')!(\nu_i x - \nu_i' y + \nu_i'')!} z^y.$$

Theorem. Assume T is generic proper hypergeom. . Then the order of a minimal telescoper for T w.r.t. y is no more than

$$B_{AZ} = \max\left\{\sum_{i=1}^{m}(\alpha_i' + \nu_i'), \sum_{i=1}^{m}(\beta_i' + \mu_i')\right\}.$$

Apagodu-Zeilberger upper bound (2005)

Definition. A hypergeom. term T is said to be proper if

$$T = p(x,y)\prod_{i=1}^{m}\frac{(\alpha_i x + \alpha_i' y + \alpha_i'')!(\beta_i x - \beta_i' y + \beta_i'')!}{(\mu_i x + \mu_i' y + \mu_i'')!(\nu_i x - \nu_i' y + \nu_i'')!} z^y.$$

$\nexists 1 \le i, j \le m$ s.t.
$\{\alpha_i = \mu_j \quad \& \quad \alpha_i' = \mu_j' \quad \& \quad \alpha_i'' - \mu_j'' \in \mathbb{N}\}$
$\{\beta_i = \nu_j \quad \& \quad \beta_i' = \nu_j'' \quad \& \quad \beta_i'' - \nu_j'' \in \mathbb{N}\}$.

Theorem. Assume T is generic proper hypergeom. . Then the order of a minimal telescoper for T w.r.t. y is no more than

$$B_{AZ} = \max\left\{\sum_{i=1}^{m}(\alpha_i' + \nu_i'), \sum_{i=1}^{m}(\beta_i' + \mu_i')\right\}.$$

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New}	B_{AZ}	$\leq B_{New}$
non-proper			
example			

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper			
example			

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\parallel$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper	B_{New}		
example			

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper	B_{New}	?	
example			

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper	B_{New}	?	$\leq B_{New}$
T_1	9	10	9

Example.

$$T_1 = \frac{(x+3y)!(x-3y)!}{(5x+3y)(3x-y)(4x-3y)!(5x+3y)!}.$$

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\|\|$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper	B_{New}	?	$\leq B_{New}$
T_2	β	$\alpha+\beta$	β

Example.

$$T_2 = \frac{\alpha^2 y^2 + \alpha^2 y - \alpha\beta y + 2\alpha x y + x^2}{(x + \alpha y + \alpha)(x + \alpha y)(x + \beta y)}, \quad \alpha \neq \beta \text{ in } \mathbb{N} \setminus \{0\}.$$

New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	B_{New} $\parallel$ $B_{AZ} - [\![\deg_y(K_n - K_d) \leq \deg_y(K_n) - 1]\!]$	B_{AZ}	$\leq B_{New}$
non-proper	B_{New}	?	$\leq B_{New}$
T_3	3	?	3

Example.

$$T_3 = \frac{x^4 + x^3y + 2x^2y^2 + 2x^2y + xy^2 + 2y^3 + x^2 + y^2 - x - y}{(x^2 + y + 1)(x^2 + y)(x + 2y)} y!$$

New lower bound

Theorem. Assume T has initial reduction

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{K_d}\right) H,$$

with b_0 integer-linear. Then the order of a telescoper for T w.r.t. y is at least

$$\max_{\substack{p \,\mid\, b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \quad \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \right\}.$$

Abramov-Le lower bound (2005)

Theorem. Assume T has initial reduction

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{K_d}\right) H = \Delta_y\Big(\ldots\Big) + \frac{a_0'}{b_0} H',$$

with b_0 integer-linear, $a_0' = a_0 K_d + b_0 q_0$ and $H' = H/K_d$. Let

$$\frac{c'}{d'} := \frac{\sigma_x(H')}{H'}.$$

Then the order of the minimal telescoper for T w.r.t. y is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \quad \min_{h\in\mathbb{Z}} \left\{ \rho \in \mathbb{N}\setminus\{0\} : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$$

New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$	$\geq \dots$
example			

Example.

New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$	$\geq \dots$
T_1	7	3	17

Example.

$$T_1 = \frac{1}{(x+3y+1)(5x-7y)(5x-7y+14)!}.$$

New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \ \min_h$ $\left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$	$\geq \dots$
T_2	12	3	29

Example.

$$T_2 = \frac{1}{(x+5y+1)(5x-12y)(5x-12y+24)!}.$$

New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$	$\max_p \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$	$\geq \dots$
T_3	α	2	α

Example.

$$T_3 = \frac{1}{(x - \alpha y - \alpha)(x - \alpha y - 2)!}, \quad \alpha \geq 2 \text{ in } \mathbb{N}.$$

Content

1. Introduction

2. Preliminaries

3. Sum decomposition for hypergeometric terms

4. Reduction-based creative telescoping

5. Upper and lower bounds

6. Summary

Summary

Results.

- Modified Abramov-Petkovšek reduction
- A reduction-based telescoping method
- Order bounds for minimal telescopers

Summary

Results.

- Modified Abramov-Petkovšek reduction
- A reduction-based telescoping method
- Order bounds for minimal telescopers

Future work.

- Complexity analysis for algorithms

Summary

Results.

- Modified Abramov-Petkovšek reduction
- A reduction-based telescoping method
- Order bounds for minimal telescopers

Future work.

- Complexity analysis for algorithms
- Reduction-based telescoping for q-hypergeometric terms

Summary

Results.

- Modified Abramov-Petkovšek reduction
- A reduction-based telescoping method
- Order bounds for minimal telescopers

Future work.

- Complexity analysis for algorithms
- Reduction-based telescoping for q-hypergeometric terms
- Creative telescoping in multivariate case

Summary

Results.

- Modified Abramov-Petkovšek reduction
- A reduction-based telescoping method
- Order bounds for minimal telescopers

Future work.

- Complexity analysis for algorithms
- Reduction-based telescoping for q-hypergeometric terms
- Creative telescoping in multivariate case

Thank you!