Efficient q-Integer Linear Decomposition of Multivariate Polynomials

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Joint work with Mark Giesbrecht, George Labahn and Eugene Zima

Outline

- Bivariate polynomials
- Multivariate polynomials

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Notation.

- R, a UFD with $\mathrm{char}(R) = 0$;
- $q \in R$, invertible and not a root of unity.

Bivariate integer-linearity

Definition. $p \in R[n, k]$ irreducible, is integer-linear over R if

$$p = P(\lambda n + \mu k),$$

where $P \in R[z]$ and $(\lambda, \mu) \in \mathbb{Z}^2$.

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Definition. $p \in R[n, k]$ is integer-linear over R if all its irreducible factors are integer-linear.

Bivariate q-integer linearity

Definition. $p \in R[q^n, q^k]$ irreducible, is q-integer linear over R if

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where $\alpha, \beta \in \mathbb{N}$, $P \in R[z]$ and $(\lambda, \mu) \in \mathbb{Z}^2$ with

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with $P(z) = 4z^2 + 2$.

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Example. $p = (2x^4 + 4y^6)y^3(x^2y^6 + 1)(xy^3 + 2)$.

$$p = x^4\, y^3\, P_1(x^{-2}y^3)\, P_2(xy^3)\, P_3(xy^3)$$

with $P_1(z) = 4z^2 + 2, P_2(z) = z^2 + 1, P_3(z) = z + 2$.

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Bivariate q-integer linear decompositions

Definition. $p \in R[x,y]$ admits *the* q-integer linear decomposition

$$p = c\, x^{\alpha} y^{\beta} P_0(x,y) \prod_{i=1}^{m} P_i(x^{\lambda_i} y^{\mu_i}),$$

where $c \in R$, $\alpha, \beta \in \mathbb{N}$, $P_0 \in R[x,y]$, $P_i \in R[z]$ and $\lambda_i, \mu_i \in \mathbb{Z}$ with

- P_0 primitive and having no non-constant q-integer linear factors;
- each P_i non-constant, primitive and $P_i(0) \neq 0$;
- $\gcd(\lambda_i, \mu_i) = 1$, and either $(\lambda_i, \mu_i) = (1, 0)$ or $\mu_i > 0$;
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q-integer linear types

Applications

- q-Integer linearity
  - q-Analogue of the Ore-Sato theorem (Du, Li 2019)
  - q-Analogue of Wilf-Zeilberger's conjecture (Chen, Koutschan 2019)
  - Applicability of q-Zeilberger's algorithm (Chen, Hou, Mu 2005)

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- q-Integer linear decomposition
  - q-Analogue of the Ore-Sato decomposition
  - Creative telescoping algorithm

A resultant-based algorithm (Le 2001)

Goal. Given $p \in R[x, y]$, find $p = c\, x^{\alpha} y^{\beta} P_0(x, y) \prod_{i=1}^{m} P_i(x^{\lambda_i} y^{\mu_i})$.

Key observation.

$$r = -\lambda_i/\mu_i \quad \Longleftrightarrow \quad \gcd(p, p(qx, q^r y)) \notin R$$

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Main steps.

- Find candidates for the (λ_i, μ_i):

$$\mathrm{cont}_x\left(\mathrm{resultant}_y\left(p, p(qx, q^r y)\right)\right) = 0$$

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$$\begin{cases} t_0 = \gcd(p, p(qx, q^r y)) \\ t_i = \gcd(t_{i-1}, t_{i-1}(qx, q^r y)), i = 1, 2, \ldots \end{cases}$$

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$$P_i(z) = \mathrm{cont}_x \left(\mathrm{num} \left(p(x^{\mu_i}, z x^{-\lambda_i}) \right) \right) \Big|_{z = z^{\frac{1}{\mu_i}}}$$

A factorization-based algorithm

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$$p = x^{\alpha} y^{\beta} P(x^{\lambda} y^{\mu}) \quad \Longleftrightarrow \quad p = x^{\alpha} y^{\beta} \sum_{k=1}^{d} c_k (x^{\lambda} y^{\mu})^k$$

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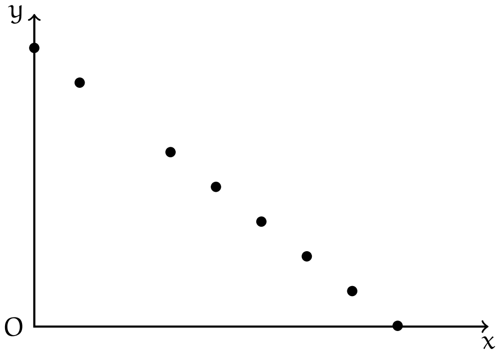

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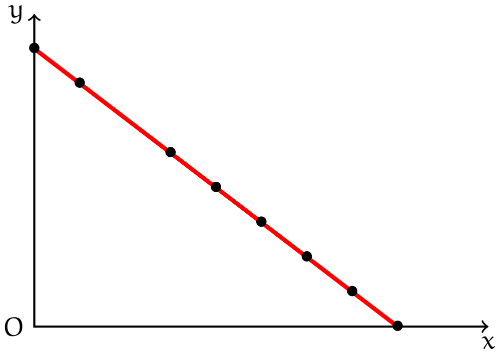

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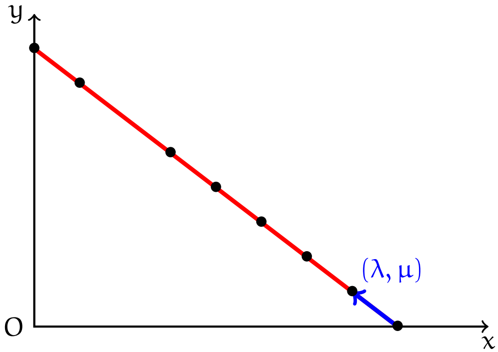

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Main steps.

- Compute the full factorization of p;
- Check the q-integer linearity of each irreducible factor;
- Group factors of the same type.

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Consider $p = 2x^4 + 4y^6$.

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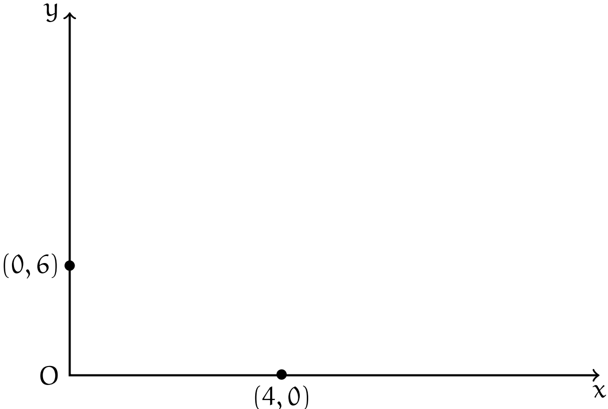

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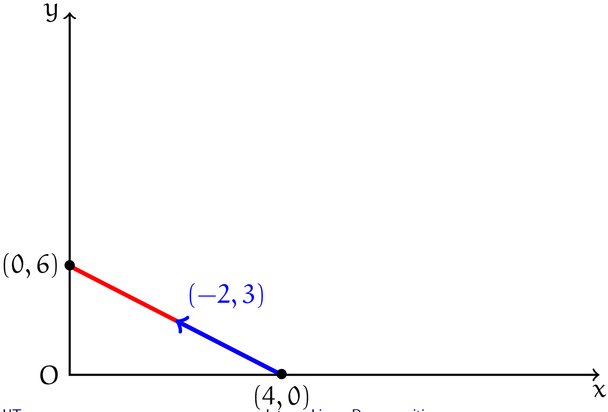

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Consider $p = (2x^4 + 4y^6)y^3(x^2y^6 + 1)(xy^3 + 2)$.

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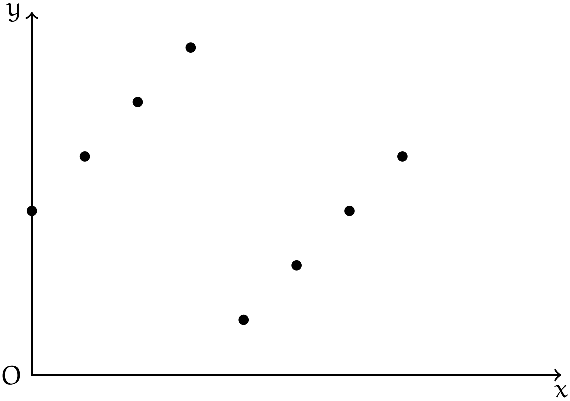

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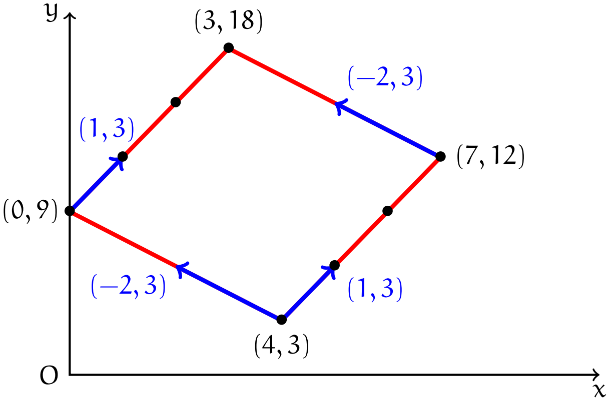

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$$p = 6x^4y^3P_1(x^{-2}y^3)P_2(x^1y^3)P_0(x, y)$$

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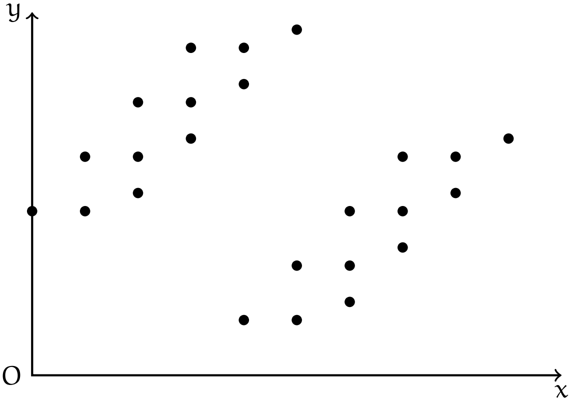

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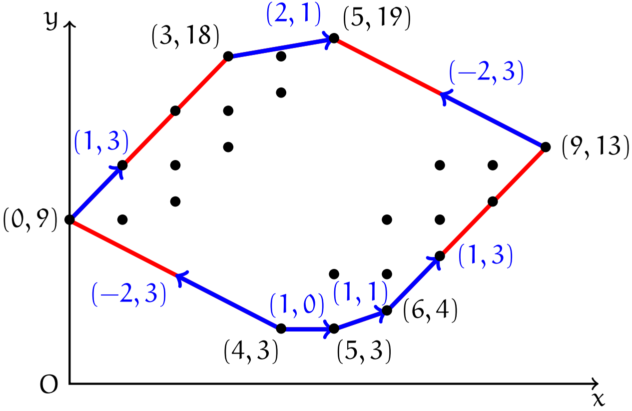

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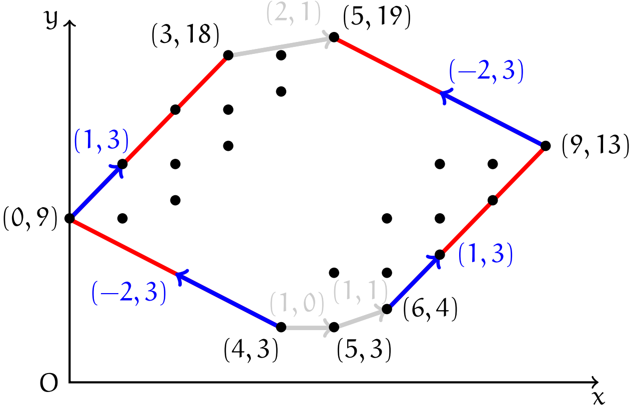

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(λ, μ) is a q-integer linear type of p

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The Newton polygon $\mathrm{Newt}(p)$ of p
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Bit complexity comparison

Given $p \in \mathbb{Z}[q, q^{-1}][x, y]$ with $\deg(p) = d$ and $\|p\|_\infty = \beta$.

BivariateQILD	ResultantQILD	FactorizationQILD
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Remark. For $p = \sum_{i,j \in \mathbb{N}, k \in \mathbb{Z}} c_{ijk} q^k x^i y^j \in \mathbb{Z}[q, q^{-1}, x, y]$,

- $\deg(p) = \max\left\{ \deg_x(p), \deg_y(p), \deg_q(p), \operatorname{ldeg}_q(p) \right\}$;
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Multivariate q-integer linear decompositions

Definition. $p \in R[x_1, \ldots, x_n]$ admits *the* q-integer linear decomposition

$$p = c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, P_0(x_1, \ldots, x_n) \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}}),$$

where $c \in R$, $\alpha_j \in \mathbb{N}$, $P_0 \in R[x_1, \ldots, x_n]$, $P_i \in R[z]$ and $\lambda_{ij} \in \mathbb{Z} \setminus \{0\}$ with

- P_0 primitive and having no non-constant q-integer linear factors;
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q-integer linear types

p is q-integer linear over R $\iff$ $P_0 = 1$

A direct idea

Given $p \in R[x_1, \ldots, x_n] \setminus R$ with $\mathrm{cont}_{x_1}(p) = \cdots = \mathrm{cont}_{x_n}(p) = 1$, find

$$p = c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, P_0(x_1, \ldots, x_n) \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}}).$$

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- Find candidates for the $(\lambda_{i1}, \ldots, \lambda_{in})$:

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- Compute the $P_i(z)$:

$$P_i(z) = \mathrm{cont}_{x_1, \ldots, x_{n-1}}\left(\mathrm{num}\left(p(x_1^{\lambda_{in}}, \ldots, x_{n-1}^{\lambda_{in}}, z x_1^{-\lambda_{i1}} \cdots x_{n-1}^{-\lambda_{i,n-1}})\right)\right)\Big|_{z = z^{\frac{1}{\lambda_{in}}}}$$

A direct idea

$\lambda_{i1}\cdots\lambda_{in}\neq 0$

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Algorithm MultivariateQILD$_1$

Input. $p \in R[x_1, \ldots, x_n]$.

Output. The q-integer linear decomposition of p.

1 If $p \in R$, return p; else $c = \mathrm{cont}_{x_1,\ldots,x_n}(p)$ and $p = p/c$.

2 If $n = 1$, return; else call the algorithm recursively on $\mathrm{cont}_{x_i}(p)$, and update $\alpha_i, P_m(x_1^{\lambda_{m1}} \cdots x_n^{\lambda_{mn}}), P_0$ and p.

3 If $p = 1$, return $c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, P_0 \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}})$.

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6 return $c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}\, P_0 \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}})$.

Example

Consider

$$(qx_1x_3+x_2^2x_3+x_2^2x_4)(7x_2^{16}x_4^{14}+2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15}-9x_1^2x_2^8x_3^3x_4^5+x_2^{12})$$

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- The support of p consists of

$$v_1 := (9,12,13,0), \quad v_2 := (8,14,13,0), \quad v_3 := (8,14,12,1),$$
$$v_4 := (11,8,16,5), \quad v_5 := (10,10,16,5), \quad v_6 := (10,10,15,6),$$
$$v_7 := (1,28,1,14), \quad v_8 := (0,30,1,14), \quad v_9 := (0,30,0,15),$$
$$v_{10} := (15,0,22,15), \quad v_{11} := (14,2,22,15), \quad v_{12} := (14,2,21,16),$$
$$v_{13} := (3,24,4,19), \quad v_{14} := (2,26,4,19), \quad v_{15} := (2,26,3,20),$$
$$v_{16} := (7,16,10,29), \quad v_{17} := (6,18,10,29), \quad v_{18} := (6,18,9,30).$$

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- The Newton polytope of p processes 19 edges:

$$\begin{array}{lllll} [v_2,v_8], & [v_8,v_{17}], & [v_{11},v_{17}], & [v_2,v_{11}], & [v_8,v_9], \\ [v_2,v_3], & [v_3,v_9], & [v_9,v_{18}], & [v_{17},v_{18}], & [v_1,v_3], \\ [v_1,v_2], & [v_{10},v_{11}], & [v_{10},v_{16}], & [v_{16},v_{17}], & [v_1,v_7], \\ [v_7,v_9], & [v_1,v_{10}], & [v_7,v_{16}], & [v_{16},v_{18}]. & \end{array}$$

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- The corresponding directions are

$$\begin{array}{lllll} (-4,8,-6,7), & (2,-4,3,5), & (-4,8,-6,7), & (2,-4,3,5), & (0,0,-1,1), \\ (0,0,-1,1), & (-4,8,-6,7), & (2,-4,3,5), & (0,0,-1,1), & (-1,2,-1,1), \\ (-1,2,0,0), & (-1,2,0,0), & (-4,8,-6,7), & (-1,2,0,0), & (-4,8,-6,7), \\ (-1,2,-1,1), & (2,-4,3,5), & (2,-4,3,5), & (-1,2,-1,1). & \end{array}$$

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$$\underbrace{(qx_1x_3 + x_2^2x_3 + x_2^2x_4)(7x_2^{16}x_4^{14} + 2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15} - 9x_1^2x_2^8x_3^3x_4^5 + x_2^{12})}_{p \in \mathbb{Z}[q,q^{-1}][x_1,x_2,x_3,x_4]}$$

Candidates for types
$(-4, 8, -6, 7)$
$(2, -4, 3, 5)$
$(-1, 2, -1, 1)$

Example

Consider

$$\underbrace{(qx_1x_3 + x_2^2x_3 + x_2^2x_4)(7x_2^{16}x_4^{14} + 2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15} - 9x_1^2x_2^8x_3^3x_4^5 + x_2^{12})}_{p \in \mathbb{Z}[q,q^{-1}][x_1,x_2,x_3,x_4]}$$

Candidates for types	Univariate polynomials
$(-4, 8, -6, 7)$	
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$(-1, 2, -1, 1)$	

$$\left.\mathrm{cont}_{x_1,\ldots,x_{n-1}}\left(\mathrm{num}\left(p(x_1^{\lambda_n},\ldots,x_{n-1}^{\lambda_n}, zx_1^{-\lambda_1}\cdots x_{n-1}^{-\lambda_{n-1}})\right)\right)\right|_{z=z^{\frac{1}{\lambda_n}}}$$

Example

Consider

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$(-4, 8, -6, 7)$	$P_1(z) = 7z^2 + 2q$
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$$\mathrm{cont}_{x_1,\ldots,x_{n-1}}\left(\mathrm{num}\left(p(x_1^{\lambda_n}, \ldots, x_{n-1}^{\lambda_n}, z x_1^{-\lambda_1} \cdots x_{n-1}^{-\lambda_{n-1}})\right)\right)\Big|_{z=z^{\frac{1}{\lambda_n}}}$$

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Example

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$$\underbrace{(qx_1x_3 + x_2^2x_3 + x_2^2x_4)(7x_2^{16}x_4^{14} + 2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15} - 9x_1^2x_2^8x_3^3x_4^5 + x_2^{12})}_{p \in \mathbb{Z}[q,q^{-1}][x_1,x_2,x_3,x_4]}$$

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- The q-integer linear decomposition of p is

$$p = x_1^8x_2^{12}x_3^{12}\, P_0\, P_1(x_1^{-4}x_2^8x_3^{-6}x_4^7)\, P_2(x_1^2x_2^{-4}x_3^3x_4^5),$$

with $P_0 = qx_1x_3 + x_2^2x_3 + x_2^2x_4$.

A proposition

Let $p \in R[x_1, \ldots, x_n]$. Then

$$p = x_1^{\alpha_1} \cdots x_n^{\alpha_n} P(x_1^{\lambda_1} \cdots x_n^{\lambda_n})$$

$$\Updownarrow$$

$$p = x_i^{\beta_{ij}} x_j^{\beta_{ji}} P_{ij}(x_i^{\mu_{ij}} x_j^{\mu_{ji}}) \quad \text{for any } 1 \le i < j \le n$$

where

- $P(z) \in R[z]$, $\alpha_i \in \mathbb{N}$, $\lambda_i \in \mathbb{Z}$;
- $P_{ij}(z) \in R[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n][z]$;
- $\beta_{ij}, \beta_{ji}, \mu_{ij}, \mu_{ji} \in \mathbb{Z}$.

A proposition

Let $p \in R[x_1, \ldots, x_n]$. Then

p is q-integer linear w.r.t. $x_1, \ldots, x_n$

$$p = x_1^{\alpha_1} \cdots x_n^{\alpha_n} P(x_1^{\lambda_1} \cdots x_n^{\lambda_n})$$

$$\Updownarrow$$

$$p = x_i^{\beta_{ij}} x_j^{\beta_{ji}} P_{ij}(x_i^{\mu_{ij}} x_j^{\mu_{ji}}) \quad \text{for any } 1 \le i < j \le n$$

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- $P(z) \in R[z]$, $\alpha_i \in \mathbb{N}$, $\lambda_i \in \mathbb{Z}$;
- $P_{ij}(z) \in R[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n][z]$;
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$$\underbrace{(qx_1x_3 + x_2^2x_3 + x_2^2x_4)(7x_2^{16}x_4^{14} + 2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15} - 9x_1^2x_2^8x_3^3x_4^5 + x_2^{12})}_{p \in \mathbb{Z}[q,q^{-1}][x_1,x_2,x_3,x_4]}$$

Example

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- $p \in \mathbb{Z}[q, q^{-1}, x_3, x_4][x_1, x_2]$

$$(qx_1x_3 + x_2^2x_3 + x_2^2x_4)(7x_2^{16}x_4^{14} + 2qx_1^8x_3^{12})(3qx_1^6x_3^9x_4^{15} - 9x_1^2x_2^8x_3^3x_4^5 + x_2^{12})$$

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Example

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$$P_0'(z, x_3) = qx_3 + zx_3 + zx_4,$$
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- $p = x_2^{28}\, P_0\, P_1'(x_1^2x_2^{-4}x_3^3)$ with

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- $P_1'(z) \in \mathbb{Z}[q, q^{-1}][z, x_4]$

$$(7x_4^{14} + 2qz^4)(3qz^3x_4^{15} - 9zx_4^5 + 1)$$

Example

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- $P_1'(z) = z^4\, P_1(z^{-2}x_4^7)\, P_2(zx_4^5)$ with

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- $p = x_1^8x_2^{12}x_3^{12}\, P_0\, P_1(x_1^{-4}x_2^8x_3^{-6}x_4^7)\, P_2(x_1^2x_2^{-4}x_3^3x_4^5)$ with

$$P_0 = qx_1x_3 + x_2^2x_3 + x_2^2x_4,$$
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Example

Consider

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$$P_1(z) = 7z^2 + 2q, \quad P_2(z) = 3qz^3 - 9z + 1$$

Algorithm MultivariateQILD$_2$

Input. $p \in R[x_1, \ldots, x_n]$.

Output. The q-integer linear decomposition of p.

1 If $p \in R$, return p; else $c = \mathrm{cont}_{x_1,\ldots,x_n}(p)$ and $p = p/c$.

2 If $n = 1$, return. If $n = 2$, call **BivariateQILD** on p and return.

3 Call algorithm recursively on $\mathrm{cont}_{x_1,x_2}(p)$ and update α_i, P_i, p.

4 If $p = 1$, return $c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n} P_0 \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}})$.

5 Set $\Lambda_1 = \{((1), p(z, x_2, \ldots, x_n))\}$ with z an indeterminate.

6 For $k = 1, \ldots, n-1$ and $((\mu_1, \ldots, \mu_k), h(z, x_{k+1}, \ldots, x_n)) \in \Lambda_k$, call **BivariateQILD** with input $h(z, x_{k+1})$ and update $\alpha_i, P_0, \Lambda_{k+1}$.

7 For $((\mu_1, \ldots, \mu_n), h(z)) \in \Lambda_n$, update $P_m(x_1^{\lambda_{m1}} \cdots x_n^{\lambda_{mn}})$.

8 return $c\, x_1^{\alpha_1} \cdots x_n^{\alpha_n} P_0 \prod_{i=1}^{m} P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}})$.

Bit complexity analysis

Given $p \in \mathbb{Z}[q, q^{-1}][x_1, \ldots, x_n]$ with $\deg(p) = d$ and $\|p\|_\infty = \beta$.

MultivariateQILD$_1$	MultivariateQILD$_2$
$O^\sim\left(n!(d^{2n+3} + d^{2n+2}\log\beta + d^{n\lfloor n/2\rfloor})\right)$	$O^\sim\left(d^{n+5} + d^{n+4}\log\beta\right)$

Remark. $p = \sum_{i_1,\ldots,i_n \in \mathbb{N}, k \in \mathbb{Z}} c_{i_1,\ldots,i_n,k} q^k x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Z}[q, q^{-1}, x_1, \ldots, x_n]$,

- $\deg(p) = \max\left\{\deg_{x_1}(p), \ldots, \deg_{x_n}(p), \deg_q(p), \operatorname{ldeg}_q(p)\right\}$;
- $\|p\|_\infty = \max_{i_1,\ldots,i_n,k} |c_{i_1,\ldots,i_n,k}|$.

Timings (in seconds)

Test suite: $p = P_0 \prod_{i=1}^{m} \mathrm{num}(P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}}))$.

- $P_0 \in \mathbb{Z}[q][x_1, \ldots, x_n]$ with $\deg_{x_1,\ldots,x_n}(P_0) = \deg_q(P_0) = d_0$;
- $(\lambda_{i1}, \ldots, \lambda_{in}) \in \mathbb{Z}^n$ with $|\lambda_{ij}| < 10$;
- $P_i(z) = f_{i1}(z) f_{i2}(z)$ with $f_{ij}(z) \in \mathbb{Z}[q][z]$ and $\deg(f_{ij}(z)) = j \cdot d$.

(n, m, d_0, d)	RQILD	FQILD	MQILD_1	MQILD_2
$(2, 1, 1, 1)$	5408.48	0.04	0.01	0.01
$(2, 1, 5, 1)$	8381.99	0.06	0.03	0.03
$(2, 1, 10, 1)$	–	0.19	0.04	0.04
$(2, 2, 10, 2)$	–	4.55	0.27	0.20
$(2, 3, 10, 2)$	–	36.14	1.38	1.21
$(2, 4, 10, 2)$	–	114.82	4.98	4.53
$(2, 3, 10, 3)$	–	169.13	4.28	3.80
$(2, 3, 10, 4)$	–	649.03	12.15	12.86
$(2, 3, 10, 5)$	–	1554.31	31.54	33.50
$(6, 2, 5, 1)$	–	1141.32	2.58	0.98
$(7, 2, 5, 1)$	–	11759.89	6.07	1.74
$(8, 2, 5, 1)$	–	18153.45	10.60	5.29
$(9, 2, 5, 1)$	–	–	65.53	38.12

Timings (in seconds)

Test suite: $p = P_0 \prod_{i=1}^{m} \text{num}(P_i(x_1^{\lambda_{i1}} \cdots x_n^{\lambda_{in}}))$.

- $P_0 \in \mathbb{Z}[q][x_1, \ldots, x_n]$ with $\deg_{x_1,\ldots,x_n}(P_0) = \deg_q(P_0) = d_0$;
- $(\lambda_{i1}, \ldots, \lambda_{in}) \in \mathbb{Z}^n$ with $|\lambda_{ij}| < 10$;
- $P_i(z) = f_{i1}(z) f_{i2}(z)$ with $f_{ij}(z) \in \mathbb{Z}[q][z]$ and $\deg(f_{ij}(z)) = j \cdot d$.

(n, m, d_0, d)	RQILD	FQILD	MQILD_1	MQILD_2
$(2, 1, 1, 1)$	5408.48	0.04	0.01	0.01
$(2, 1, 5, 1)$	8381.99	0.06	0.03	0.03
$(2, 1, 10, 1)$	–	0.19	0.04	0.04
$(2, 2, 10, 2)$	–	4.55	0.27	0.20
$(2, 3, 10, 2)$	–	36.14	1.38	1.21
$(2, 4, 10, 2)$	–	114.82	4.98	4.53
$(2, 3, 10, 3)$	–	169.13	4.28	3.80
$(2, 3, 10, 4)$	–	649.03	12.15	12.86
$(2, 3, 10, 5)$	–	1554.31	31.54	33.50
$(6, 2, 5, 1)$	–	1141.32	2.58	0.98
$(7, 2, 5, 1)$	–	11759.89	6.07	1.74
$(8, 2, 5, 1)$	–	18153.45	10.60	5.29
$(9, 2, 5, 1)$	–	–	65.53	38.12

Summary

Results.

- An efficient algorithm for bivariate q-integer linear decompositions
- Two algorithms to handle general multivariate polynomials