D-finite Numbers

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Abstract

D-finite functions and P-recursive sequences are defined in terms of linear differential and recurrence equations with polynomial coefficients. In this paper, we introduce a class of numbers closely related to D-finite functions and P-recursive sequences. It consists of the limits of convergent P-recursive sequences. Typically, this class contains many well-known mathematical constants in addition to the algebraic numbers. Our definition of the class of D-finite numbers depends on two subrings of the field of complex numbers. We investigate how different choices of these two subrings affect the class. Moreover, we show that D-finite numbers are essentially limits of D-finite functions at the point one, and evaluating D-finite functions at non-singular algebraic points typically yields D-finite numbers. This result makes it easier to recognize certain numbers to be D-finite.

1 Introduction

D-finite functions have been recognized long ago [23, 15, 30, 19, 16, 24] as an especially attractive class of functions. They are interesting on the one hand because each of them can be easily described by a finite amount of data, and efficient algorithms are available to do exact as well as approximate computations with them. On the other hand, the class is interesting because it covers a lot of special functions which naturally appear in various different context, both within mathematics as well as in applications.

The defining property of a *D-finite* function is that it satisfies a linear differential equation with polynomial coefficients. This differential equation, together with an appropriate number of initial terms, uniquely determines the function at hand. Similarly, a sequence is called *P-recursive* (or rarely, *D-finite*) if it satisfies a linear recurrence equation with polynomial coefficients. Also in this case, the equation together with an appropriate number of initial terms uniquely determines the object.

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In a sense, the theory of D-finite functions generalizes the theory of algebraic functions. Many concepts that have first been introduced for the latter have later been formulated also for the former. In particular, every algebraic function is D-finite (Abel's theorem), and many properties the class of algebraic function enjoys carry over to the class of D-finite functions.

The theory of algebraic functions in turn may be considered as a generalization of the classical and well-understood class of algebraic numbers. The class of algebraic numbers suffers from being relatively small. There are many important numbers, most prominently the numbers e and π, which are not algebraic.

Many larger classes of numbers have been proposed, let us just mention three examples. The first is the class of periods (in the sense of Kontsevich and Zagier [14]). These numbers are defined as the values of multivariate definite integrals of algebraic functions over a semi-algebraic set. In addition to all the algebraic numbers, this class contains important numbers such as π, all zeta constants (the Riemann zeta function evaluated at an integer) and multiple zeta values, but it is so far not known whether for example e, $1/\pi$ or Euler's constant γ are periods (conjecturally they are not). The second example is the class of all numbers that appear as values of so-called G-functions (in the sense of Siegel [21]) at algebraic number arguments [4, 5]. The class of G-functions is a subclass of the class of D-finite functions, and it inherits some useful properties of that class. Among the values that G-functions can assume are π, $1/\pi$, values of elliptic integrals, and multiple zeta values, but it is so far not known whether for example e, Euler's constant γ or a Liouville number are such a value (conjecturally they are not).

Another class of numbers is the class of holonomic constants, studied by Flajolet and Vallée [9, §4]. (We thank Marc Mezzarobba for pointing us to this reference.) A constant that is the value $f(z_0)$ of a D-finite function $f(z)$ at an algebraic point z_0 where $f(z)$ is regular (i.e., analytic) is called a *regular holonomic constant*. Classical examples are π, $\log(2)$ and the polylogarithm value $\mathrm{Li}_4(1/2)$. A *singular holonomic constant* is defined to be the value of a D-finite function $f(z)$ at a Fuchsian singularity (also known as regular singularity [29]) of a defining differential equation for $f(z)$. Note that the classes of regular and singular holonomic constants are not completely opposite to each other, since a constant can be of both types. A typical example is Apéry's constant $\zeta(3)$. This constant is of singular type since $\zeta(3) = \mathrm{Li}_3(1)$ where the polylogarithm function $\mathrm{Li}_3(z)$ is D-finite and has a singularity at one of the Fuchsian type. On the other hand, $\zeta(3)$ is also a regular holonomic constant, because $\mathrm{Li}_3(z)$ is a G-function and values of G-functions at algebraic numbers are all of regular type by [4, Theorem 1].

It is tempting to believe that there is a strong relation between holonomic constants and limits of convergent P-recursive sequences. To make this relation precise, we introduce the class of D-finite numbers in this paper.

Definition 1. *Let R be a subring of $\mathbb{C}$ and let $\mathbb{F}$ be a subfield of $\mathbb{C}$.*

1. *A number $\xi \in \mathbb{C}$ is called* D-finite *(with respect to R and $\mathbb{F}$) if there exists a convergent sequence $(a_n)_{n=0}^{\infty}$ in $R^{\mathbb{N}}$ with $\lim_{n\to\infty} a_n = \xi$ and some polynomials $p_0, \dots, p_r \in \mathbb{F}[n]$, $p_r \neq 0$, such that*
$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$
for all $n \in \mathbb{N}$.

2. *The set of all D-finite numbers with respect to R and $\mathbb{F}$ is denoted by $\mathcal{D}_{R,\mathbb{F}}$. If $R = \mathbb{F}$, we also write $\mathcal{D}_{\mathbb{F}} := \mathcal{D}_{\mathbb{F},\mathbb{F}}$ for short.*

It is clear that $\mathcal{D}_{R,\mathbb{F}}$ contains all the elements of R, but it typically contains many further elements. For example, let i be the imaginary unit, then $\mathcal{D}_{\mathbb{Q}(i)}$ contains many (if not all) the periods and, as we will see below, all the values of G-functions as well as many (if not all) regular holonomic constants. In addition, it is not hard to see that e and $1/\pi$ are D-finite numbers. According to Fischler and Rivoal's work [5], also Euler's constant γ and any value of the Gamma function at a rational number are D-finite. (We thank Alin Bostan for pointing us to this reference.) We will show below that D-finite numbers are essentially the limiting values of D-finite functions at one. Moreover, the values D-finite functions can assume at non-singular algebraic points are in fact D-finite numbers. Together with the work on arbitrary-precision evaluation of D-finite functions [3, 25, 26, 27, 17, 18], it follows that D-finite numbers are computable in the sense that for every D-finite number ξ there exists an algorithm which for any given $n \in \mathbb{N}$ computes a numeric approximation of ξ with a guaranteed precision of 10^{-n}. Consequently, all non-computable numbers have no chance to be D-finite. Besides these artificial examples, we do not know of any explicit real numbers which are not in $\mathcal{D}_{\mathbb{Q}}$, and we believe that it may be very difficult to find some.

The definition of D-finite numbers given above involves two subrings of $\mathbb{C}$ as parameters: the ring to which the sequence terms of the convergent sequences are supposed to belong, and the field to which the coefficients of the polynomials in the recurrence equations should belong. Obviously, these choices matter, because we have, for example, $\mathcal{D}_{\mathbb{R}} = \mathbb{R} \neq \mathbb{C} = \mathcal{D}_{\mathbb{C}}$. Also, since $\mathcal{D}_{\mathbb{Q}}$ is a countable set, we have $\mathcal{D}_{\mathbb{Q}} \neq \mathcal{D}_{\mathbb{R}}$. On the other hand, different choices of R and $\mathbb{F}$ may lead to the same classes. For example, we would not get more numbers by allowing $\mathbb{F}$ to be a subring of $\mathbb{C}$ rather than a field, because we can always clear denominators in a defining recurrence. One of the goals of this article is to investigate how R and $\mathbb{F}$ can be modified without changing the resulting class of D-finite numbers.

As a long-term goal, we hope to establish the notion of D-finite numbers as a class that naturally relates to the class of D-finite functions in the same way as the classical class of algebraic numbers relates to the class of algebraic functions.

2 D-finite Functions and P-recursive Sequences

Throughout the paper, R is a subring of $\mathbb{C}$ and $\mathbb{F}$ is a subfield of $\mathbb{C}$, as in Definition 1 above. We consider linear operators that act on sequences or power series and analytic functions. We write S_n for the shift operator w.r.t. n which maps a sequence $(a_n)_{n=0}^{\infty}$ to $(a_{n+1})_{n=0}^{\infty}$. The set of all linear operators of the form $L := p_0 + p_1 S_n + \cdots + p_r S_n^r$, with $p_0, \ldots, p_r \in \mathbb{F}[n]$, forms an Ore algebra; we denote it by $\mathbb{F}[n]\langle S_n \rangle$. Analogously, we write D_z for the derivation operator w.r.t. z which maps a power series or function $f(z)$ to its derivative $f'(z) = \frac{d}{dz} f(z)$. Also the set of linear operators of the form $L := p_0 + p_1 D_z + \cdots + p_r D_z^r$, with $p_0, \ldots, p_r \in \mathbb{F}[z]$, forms an Ore algebra; we denote it by $\mathbb{F}[z]\langle D_z \rangle$. For an introduction to Ore algebras and their actions, see [1]. When $p_r \neq 0$, we call r the *order* of the operator and $\operatorname{lc}(L) := p_r$ its *leading coefficient.*

Definition 2.

1. *A sequence* $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ *is called* P-recursive *or* D-finite *over* $\mathbb{F}$ *if there exists a nonzero operator* $L = \sum_{j=0}^{r} p_j(n) S_n^j \in \mathbb{F}[n]\langle S_n \rangle$ *such that*

$$L \cdot a_n = p_r(n) a_{n+r} + \cdots + p_1(n) a_{n+1} + p_0(n) a_n = 0$$

for all $n \in \mathbb{N}$.

2. *A formal power series $f(z) \in R[[z]]$ is called* D-finite *over $\mathbb{F}$ if there exists a nonzero operator $L = \sum_{j=0}^{r} p_j(z) D_z^j \in \mathbb{F}[z]\langle D_z\rangle$ such that*

$$L \cdot f(z) = p_r(z) D_z^r f(z) + \cdots + p_1(z) D_z f(z) + p_0(z) f(z) = 0.$$

3. *An analytic function $f\colon U \to \mathbb{C}$ defined in some open set $U \subseteq \mathbb{C}$ is called* D-finite *over $\mathbb{F}$ if there exists a nonzero operator $L = \sum_{j=0}^{r} p_j(z) D_z^j \in \mathbb{F}[z]\langle D_z\rangle$ such that*

$$L \cdot f(z) = p_r(z) D_z^r f(z) + \cdots + p_1(z) D_z f(z) + p_0(z) f(z) = 0$$

for all $z \in U$.

4. *A formal power series $f(z) \in \mathbb{F}[[z]]$ is called* algebraic *over $\mathbb{F}$ if there exists a nonzero bivariate polynomial $P(z, y) \in \mathbb{F}[z, y]$ such that $P(z, f(z)) = 0$.*

5. *An analytic function $f\colon U \to \mathbb{C}$ defined in some open set $U \subseteq \mathbb{C}$ is called* algebraic *over $\mathbb{F}$ if there exists a nonzero bivariate polynomial $P(z, y) \in \mathbb{F}[z, y]$ such that $P(z, f(z)) = 0$ for all $z \in U$.*

Unless there is a danger of confusion, we will not strictly distinguish between complex functions that are analytic in some neighborhood of zero and the formal power series appearing as their Taylor expansions at zero. In particular, if the formal power series $f \in \mathbb{F}[[z]]$ happens to be convergent, we may as well regard it as an analytic function defined in some open neighborhood of zero.

A formal power series (or function) is D-finite if and only if its coefficient sequence is P-recursive. Many functions like exponentials, logarithms, sine, arcsine and hypergeometric series, as well as many formal power series like $\sum n! z^n$, are D-finite. Hence their respective coefficient sequences are P-recursive.

The class of D-finite functions (resp. P-recursive sequences) is closed under certain operations: addition, multiplication, derivative (resp. forward shift) and integration (resp. summation). In particular, the set of D-finite functions (resp. P-recursive sequences) forms a left-$\mathbb{F}[z]\langle D_z\rangle$-module (resp. a left-$\mathbb{F}[n]\langle S_n\rangle$-module). Also, if f is a D-finite function and g is an algebraic function, then the composition $f \circ g$ is D-finite. These and further closure properties are easily proved by linear algebra arguments, proofs can be found for instance in [23, 19, 12]. We will make free use of these facts.

We will be considering singularities of D-finite functions. Recall from the classical theory of linear differential equations [11] that a linear differential equation $p_0(z)f(z) + \cdots + p_r(z)f^{(r)}(z) = 0$ with polynomial coefficients $p_0, \ldots, p_r \in \mathbb{F}[z]$ and $p_r \neq 0$ has a basis of analytic solutions in a neighborhood of every point $\xi \in \mathbb{C}$, except possibly at roots of p_r. The roots of p_r are therefore called the *singularities* of the equation (or the corresponding linear operator). All other points are *ordinary points* (or *non-singular points*) of the equation. The behaviour of a D-finite function near a singularity z_0 can in general not be described by a formal power series, but it is always a linear combination of generalized series of the form

$$\exp(P((z - z_0)^{-1/s}))(z - z_0)^{\alpha} a\big((z - z_0)^{1/s}, \log(z - z_0)\big)$$

for some $s \in \mathbb{N}$, $P \in \bar{\mathbb{F}}[z]$, $\alpha \in \bar{\mathbb{F}}$, and $a \in \bar{\mathbb{F}}[[x]][y]$. See [11] for details of this construction. Formal power series are in general also not sufficient to describe the behaviour for algebraic functions, but such functions are always linear combinations of so-called Puiseux series, which can be written in the form

$$\sum_{n=n_0}^{\infty} a_n(z-z_0)^{n/s}$$

for some $n_0 \in \mathbb{Z}$ and some positive integer s. See, e.g., [28] for details.

It can happen that $\xi \in \mathbb{C}$ is a singularity of the equation but the equation nevertheless admits a basis of analytic solutions at this point. Such a singularity is called an *apparent singularity.* It is well-known [11, 2] that for any given linear differential equations with some apparent and some non-apparent singularities, we can always construct another linear differential equation (typically of higher order) whose solution space contains the solution space of the first equation and whose only singularities are the non-apparent singularities of the first equation. This process is known as desingularization. For later use, we will give a proof of the composition closure property for D-finite functions which pays attention to the singularities.

Theorem 3. *Let $P(z,y) \in \mathbb{F}[z,y]$ be a polynomial of degree d in y, and let $L \in \mathbb{F}[z]\langle D_z\rangle$ nonzero. Let $\zeta \in \mathbb{C}$ be such that P defines d distinct analytic algebraic functions $g(z)$ with $P(z,g(z)) = 0$ in a neighborhood Ω of ζ, and assume that for none of these functions, the value $g(\zeta) \in \mathbb{C}$ is a singularity of L. Fix a solution g of P and an analytic solution f of L defined in a neighborhood of $g(\zeta)$. Then there exists a nonzero operator $M \in \mathbb{F}[z]\langle D_z\rangle$ with $M \cdot (f \circ g) = 0$ which does not have ζ among its singularities. Moreover, any point in the neighborhood Ω with the property that none of the evaluations at this point of the d solutions of P near ζ gives a singularity of L, is an ordinary point of M.*

Proof. (borrowed from [13]) Let g be a root of P near ζ. If g is constant, then so is $f \circ g$ and we can take $M = D_z$. Suppose that g is not constant. Without loss of generality, we may assume that P is irreducible (if it is not, replace P by the minimal polynomial of g). Then none of the solutions of P is constant.

Consider the operator $\tilde{L} := L(g,(g')^{-1}D_z) \in \overline{\mathbb{F}(z)}\langle D_z\rangle$. Because of $D_z \cdot (f \circ g) = (f' \circ g)g'$, we have $L \cdot f = 0$ if and only if $\tilde{L} \cdot (f \circ g) = 0$. Therefore, if $f_1,\dots,f_r$ is a basis of the solution space of L near $g(\zeta)$, then $f_1 \circ g,\dots,f_r \circ g$ is a basis of the solution space of $\tilde{L}$ near ζ.

Let $g_1,\dots,g_d$ be all the solutions of P near ζ, and let M be the least common left multiple of all the operators $L(g_j,(g_j')^{-1}D_z)$. Then the solution space of M near ζ is generated by all the functions $f_i \circ g_j$. The Galois group G of P consists of all automorphisms of the field $K = \mathbb{F}(z,g_1,\dots,g_d)$ which leave $\mathbb{F}(z)$ fixed. The Galois group respects the differential structure of the field K in the sense that for all $u \in K$ and all $\pi \in G$ we have $\pi(u') = \pi(u)'$. Therefore, the action of G on K naturally extends to an action of G on the ring $K\langle D_z\rangle$ of linear differential operators. Since M is the least common left multiple of all the operators $L(g_j,(g_j')^{-1}D_z)$, regardless of their order, we have $\pi(M) = M$ for all $\pi \in G$. This implies that $M \in \mathbb{F}(z)\langle D_z\rangle$. (This argument already appears in Section 61 of [20].)

After clearing denominators (from the left) if necessary, we may assume that M is an operator in $\mathbb{F}[z]\langle D_z\rangle$ whose solution space is generated by functions that are analytic at ζ. Since g_j are analytic at any point η in the neighborhood Ω, the functions that generate the solution space of M are also analytic at $\eta \in \Omega$ provided that none of the values $g_j(\eta)$ is a singularity of L. Therefore, by the remarks made about desingularization, it is possible to replace M by an operator (possibly of higher order) which does not have ζ and such η among its singularities. $\square$

By a similar argument, we see that algebraic extensions of the coefficient field of the recurrence operators are useless. Moreover, it is also not useful to make $\mathbb{F}$ bigger than the quotient field of R.

Lemma 4.

1. *If $\mathbb{E}$ is an algebraic extension field of $\mathbb{F}$ and $(a_n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{E}$, then it is also P-recursive over $\mathbb{F}$.*

2. *If $R \subseteq \mathbb{F}$ and $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ is P-recursive over $\mathbb{F}$, then it is also P-recursive over* $\operatorname{Quot}(R)$, *the quotient field of R.*

3. *If $\mathbb{F}$ is closed under complex conjugation and $(a_n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{F}$, then so are $(\bar{a}_n)_{n=0}^{\infty}$, $(\operatorname{Re}(a_n))_{n=0}^{\infty}$, and $(\operatorname{Im}(a_n))_{n=0}^{\infty}$.*

Proof. 1. Let $L \in \mathbb{E}[n]\langle S_n\rangle$ be an annihilating operator of $(a_n)_{n=0}^{\infty}$. Then, since L has only finitely many coefficients, $L \in \mathbb{F}(\theta)[n]\langle S_n\rangle$ for some $\theta \in \mathbb{E}$. Let M be the least common left multiple of all the conjugates of L. Then M is an annihilating operator of $(a_n)_{n=0}^{\infty}$ which belongs to $\mathbb{F}[n]\langle S_n\rangle$. The claim follows.

2. Let us write $\mathbb{K} = \operatorname{Quot}(R)$. Let $L \in \mathbb{F}[n]\langle S_n\rangle$ be a nonzero annihilating operator of $(a_n)_{n=0}^{\infty}$. Since $\mathbb{F}$ is an extension field of $\mathbb{K}$, it is a vector space over $\mathbb{K}$. Write

$$L = \sum_{m=0}^{r} \sum_{j=0}^{d_m} p_{mj} n^j S_n^m,$$

where $r, d_m \in \mathbb{N}$ and $p_{mj} \in \mathbb{F}$ not all zero. Then the set of the coefficients p_{ij} belongs to a finite dimensional subspace of $\mathbb{F}$. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis of this subspace over $\mathbb{K}$. Then for each pair (m, j), there exists $c_{mj\ell} \in \mathbb{K}$ such that $p_{mj} = \sum_{\ell=1}^{s} c_{mj\ell}\alpha_\ell$, which gives

$$0 = L \cdot a_n = \sum_{\ell=1}^{s} \alpha_\ell \underbrace{\left(\sum_{m=0}^{r} \sum_{j=0}^{d_m} c_{mj\ell} n^j a_{n+m} \right)}_{=:b_n \in \mathbb{K}}.$$

For all $n \in \mathbb{N}$, it follows from the linear independence of $\{\alpha_1, \ldots, \alpha_s\}$ that $b_n = 0$. Therefore

$$\sum_{m=0}^{r} \underbrace{\left(\sum_{j=0}^{d_m} c_{mj\ell} n^j \right)}_{\in \mathbb{K}[n]} S_n^m \cdot a_n = 0 \quad \text{for all } n \in \mathbb{N} \text{ and } \ell = 1, \ldots, s.$$

Thus $(a_n)_{n=0}^{\infty}$ has a nonzero annihilating operator with coefficients in $\mathbb{K}[n]$.

3. Since $(a_n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{F}$, there exists a nonzero operator L in $\mathbb{F}[n]\langle S_n\rangle$ such that $L \cdot a_n = 0$. Hence $\bar{L} \cdot \bar{a}_n = 0$ where $\bar{L}$ is the operator obtained from L by taking the complex conjugate of each coefficient. Since $\mathbb{F}$ is closed under complex conjugation by assumption, $\bar{L}$ belongs to $\mathbb{F}[n]\langle S_n\rangle$, and hence $(\bar{a}_n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{F}$. Because of

$$\operatorname{Re}(a_n) = \frac{1}{2}(a_n + \bar{a}_n) \quad \text{and} \quad \operatorname{Im}(a_n) = \frac{1}{2i}(a_n - \bar{a}_n),$$

where i is the imaginary unit, the assertions follow by closure properties.

∎

Of course, all the statements hold analogously for D-finite formal power series instead of P-recursive sequences.

If a D-finite function is analytic in a neighborhood of zero, then it can be extended by analytic continuation to any point in the complex plane except for finitely many ones, namely the singularities of the given function. Those closest to the origin are called *dominant singularities* of the function. In this sense, D-finite functions can be evaluated at any non-singular point by means of analytic continuation. Numerical evaluation algorithms for D-finite functions have been developed in [3, 25, 26, 27, 17, 18], where the last two references also provide a Maple implementation, namely the NumGfun package, for computing such evaluations. These algorithms perform arbitrary-precision evaluations with full error control.

3 Algebraic Numbers

Before turning to general D-finite numbers, let us consider the subclass of algebraic functions. We will show that in this case, the possible limits are precisely the algebraic numbers. For the purpose of this article, let us say that a sequence $(a_n)_{n=0}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ is *algebraic* over $\mathbb{F}$ if the corresponding power series $\sum_{n=0}^{\infty} a_n z^n \in \mathbb{F}[[z]]$ is algebraic in the sense of Definition 2. Since algebraic functions are D-finite, it is clear that algebraic sequences are P-recursive. We will write $\mathcal{A}_{\mathbb{F}}$ for the set of all complex numbers which are limits of convergent algebraic sequences over $\mathbb{F}$.

Recall that two sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ with at most finitely many zero terms are called *asymptotically equivalent,* written $a_n \sim b_n$ $(n \to \infty)$, if the quotient a_n/b_n converges to one as n tends to infinity. Similarly, two complex functions $f(z)$ and $g(z)$ are called *asymptotically equivalent* at a point $\zeta \in \mathbb{C}$, written $f(z) \sim g(z)$ $(z \to \zeta)$, if the quotient $f(z)/g(z)$ converges to one as z approaches ζ. These notions are connected by the following classical theorem.

Theorem 5.

1. *(Transfer theorem [7, 8]) Assume that $f(z) \in \mathbb{F}[[z]]$ is analytic at zero with the only dominant singularity $z = 1$ and*
$$f(z) \sim \frac{1}{(1-z)^{\alpha}} \quad (z \to 1)$$
with $\alpha \notin \{0, -1, -2, \dots\}$. Then
$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad (n \to \infty),$$
where $\Gamma(z)$ stands for the Gamma function and the notation $[z^n]f(z)$ refers to the coefficient of z^n in $f(z)$.

2. *(Basic Abelian theorem [6]) Let $(a_n)_{n=0}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ be a sequence that satisfies the asymptotic estimate*
$$a_n \sim n^{\alpha} \quad (n \to \infty),$$
where $\alpha \geq 0$. Then the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies the asymptotic estimate
$$f(z) \sim \frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}} \quad (z \to 1^-).$$
This estimate remains valid when z tends to one in any sector with vertex at one symmetric about the horizontal axis, and with opening angle less than π.

Also recall the formal version of the implicit function theorem [22], which says that for any bivariate polynomial $P(z,y) \in \mathbb{F}[z,y]$ with $P(0,0) = 0$ and $(D_y P)(0,0) \neq 0$, there exists a unique formal power series $f(z) \in \mathbb{F}[[z]]$ with $f(0) = 0$ so that $P(z, f(z)) = 0$.

With the above preparations, we are ready to develop the following key lemma, which indicates that depending on whether $\mathbb{F}$ is a real field or not, every real algebraic number or every algebraic number can appear as a limit of an algebraic sequence over $\mathbb{F}$.

Lemma 6. *Let $p(y) \in \mathbb{F}[y]$ be an irreducible polynomial of degree d. Assume that $\zeta_1, \ldots, \zeta_d \in \bar{\mathbb{F}}$ are all the roots of p. Then there exists a polynomial $P(z,y) \in \bar{\mathbb{F}}[z,y]$ of degree d in y admitting d distinct roots in $\bar{\mathbb{F}}[[z]]$ such that for each $j = 1, \ldots, d$, there is exactly one $f_j(z) \in \bar{\mathbb{F}}[[z]]$ with $P(z, f_j(z)) = 0$ and $\lim_{n\to\infty}[z^n]f_j(z) = \zeta_j$. All these $f_j(z)$ are analytic at zero with the only possible dominant singularity at $z = 1$, which can at most be a simple pole. Furthermore, if*

$$\text{either} \quad (\mathbb{F} \subseteq \mathbb{R} \text{ and } \zeta_1 \in \bar{\mathbb{F}} \cap \mathbb{R}) \quad \text{or} \quad (\mathbb{F} \setminus \mathbb{R} \neq \emptyset),$$

then $P(z,y)$ can be chosen in $\mathbb{F}[z,y]$ so that $f_1(z) \in \mathbb{F}[[z]]$.

Proof. If $d = 1$, then $p(y) = y - \zeta_1$ with $\zeta_1 \in \mathbb{F}$. Letting $P(z,y) = p(y)$ yields the assertions.

Now assume that $d > 1$. Then $\zeta_j \neq 0$ for all $j = 1, \ldots, d$ since p is irreducible. Let $\varepsilon > 0$ be such that any two (real or complex) roots of p have a distance of more than 2ε to each other. Such an ε exists because p is an irreducible polynomial of degree greater than one, and thus has only finitely many distinct roots. The roots of a polynomial depend continuously on its coefficients. Therefore there exists a real number $\delta > 0$ so that perturbing the coefficients by up to δ won't perturb the roots by more than $\varepsilon/2$. Any positive smaller number than δ will have the same property. By the choice of ε, any such perturbation of the polynomial will have exactly one root in each of the open balls of radius $\varepsilon/2$ centered at the roots of p.

For fixed nonzero $\alpha \in \bar{\mathbb{F}}$ with $|\alpha| < \delta/2$, consider the perturbation $\tilde{P}_\alpha(z,y) = p(y) - \alpha(1-z) \in \mathbb{F}(\alpha)[z,y]$. We will show that

($*$) the polynomial $\tilde{P}_\alpha(z,y)$ has exactly d distinct roots in $\overline{\mathbb{F}(z)}$ for fixed z with $|z| \leq 1$, and any two of them have a distance of more than ε. Moreover, there exist functions $g_1, \ldots, g_d$ defined for $|z| \leq 1$ such that $g_j(1) = \zeta_j$ and $\tilde{P}_\alpha(z, g_j(z)) = 0$ and $|g_j(z) - \zeta_j| < \varepsilon/2$ for all z with $|z| \leq 1$ and $j = 1, \ldots, d$.

In fact, since $|\alpha| < \delta/2$, for any z in the disk $|z| \leq 1$ we have

$$|-\alpha(1-z)| \leq 2|\alpha| < \delta.$$

Therefore, for every z with $|z| \leq 1$, each root of $\tilde{P}_\alpha(z,y)$ belongs to exactly one open ball of radius $\varepsilon/2$ centered at a root ζ_j of $p(y)$, and by continuity, as z varies and tends to one in the disk $|z| \leq 1$, each root approaches the root ζ_j inside the corresponding open ball (Fig. 1).

Since any two roots of p are separated by more than 2ε, the distance between any two roots of $\tilde{P}_\alpha(z,y)$ for fixed z with $|z| \leq 1$ is more than ε. We have thus shown ($*$).

Now, let $\eta_1, \ldots, \eta_d \in \bar{\mathbb{F}}$ be the d distinct roots of $\tilde{P}_\alpha(0,y)$, and let their indexing be such that $|\eta_j - \zeta_j| < \varepsilon/2$ for each j. Note that $\tilde{P}_\alpha(0,y)$ is square-free because $|\eta_i - \eta_j| > \varepsilon$ for $i \neq j$. This means that $\gcd(\tilde{P}_\alpha(0,y), (D_y\tilde{P}_\alpha)(0,y)) = 1$. By $(D_y\tilde{P}_\alpha)(0,y) = p'(y)$, we have $p'(\eta_j) \neq 0$ for all j. It follows that

$$\tilde{P}_\alpha(0,\eta_j) = 0 \quad \text{and} \quad (D_y\tilde{P}_\alpha)(0,\eta_j) = p'(\eta_j) \neq 0.$$

Applying the implicit function theorem to each $\tilde{P}_\alpha(z, y + \eta_j) \in \mathbb{F}(\alpha,\eta_j)[z,y]$ (with $\mathbb{F}(\alpha,\eta_j)$ in place of $\mathbb{F}$) yields that there exist d distinct formal power series $g_1(z), \ldots, g_d(z)$ with each

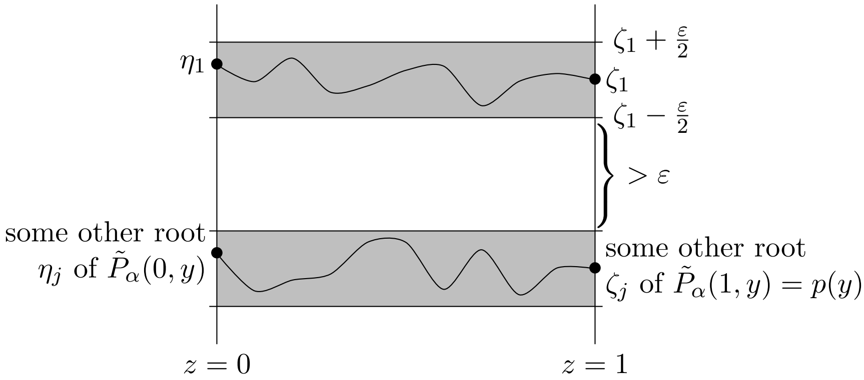

Figure 1: Separation of roots as used in the proof of Lemma 6

$g_j(z) \in \mathbb{F}(\alpha, \eta_j)[[z]]$ and $g_j(0) = \eta_j$ such that $\tilde{P}_\alpha(z, g_j(z)) = 0$. By $(*)$, for each j there exists a unique integer k with $1 \leq k \leq d$ so that $g_j(1) = \zeta_k$ and $|g_j(z) - \zeta_k| < \varepsilon/2$ for any z with $|z| \leq 1$. Hence $|\eta_j - \zeta_k| < \varepsilon/2$ since $\eta_j = g_j(0)$. By $|\eta_j - \zeta_j| < \varepsilon/2$, we get $|\zeta_j - \zeta_k| < \varepsilon$. Thus $j = k$ because any two roots of p are separated by more than 2ε.

Moreover, all $g_j(z) \in \mathbb{F}(\alpha, \eta_j)[[z]]$ annihilated by $\tilde{P}_\alpha(z, y)$ are analytic in the disk $|z| \leq 1$. Indeed, since the leading coefficient of $\tilde{P}_\alpha(z, y)$ w.r.t. y is a nonzero constant, the singularities of the $g_j(z)$ could only be branch points. However, the choices of ε and δ make it impossible for the $g_j(z)$ to have branch points in the disk $|z| \leq 1$, because in order to have a branch point, two roots of the polynomial $\tilde{P}_\alpha(z, y)$ w.r.t. y would need to touch each other as z varies, and we have ensured that they are always separated by more than ε as z ranges over the unit disk (see $(*)$ and Fig. 1).

Now define the polynomial

$$P_\alpha(z, y) = \tilde{P}_\alpha(z, (1 - z)y) = p((1 - z)y) - \alpha(1 - z) \in \mathbb{F}(\alpha)[z, y].$$

Observe that for any $g(z) \in \bar{\mathbb{F}}[[z]]$, we have $g(z)/(1 - z)$ is a root of $P_\alpha(z, y)$ if and only if $g(z)$ is a root of $\tilde{P}_\alpha(z, y)$. Thus there exist exactly d distinct formal power series

$$f_j(z) = \frac{g_j(z)}{(1 - z)} \in \mathbb{F}(\alpha, \eta_j)[[z]] \subseteq \bar{\mathbb{F}}[[z]]$$

with $f_j(0) = g_j(0) = \eta_j$ and $g_j(1) = \zeta_j$ such that $P_\alpha(z, f_j(z)) = 0$.

Since each $g_j(z)$ is analytic in the disk $|z| \leq 1$ and $g_j(1) = \zeta_j \neq 0$, the point $z = 1$ is evidently the only singularity of $f_j(z)$ in the disk $|z| \leq 1$, and thus it is the only dominant singularity. In addition, the point $z = 1$ is further a simple pole of $f_j(z)$ and then

$$f_j(z) \sim \frac{\zeta_j}{1 - z} \quad (z \to 1),$$

which gives $[z^n]f_j(z) \sim \zeta_j$ $(n \to \infty)$ by part 1 of Theorem 5 (with $\bar{\mathbb{F}}$ in place of $\mathbb{F}$). Since $\zeta_j \neq 0$, it follows that $\lim_{n\to\infty}[z^n]f_j(z) = \zeta_j$.

Further assume that either $\mathbb{F} \subseteq \mathbb{R}$ and $\zeta_1 \in \bar{\mathbb{F}} \cap \mathbb{R}$, or $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$. In either case, $\mathbb{F}$ is dense in the field $\mathbb{F}(\zeta_1)$ since $\mathbb{F} \supseteq \mathbb{Q}$. Then by the continuity of p at ζ_1, with the above δ and ε, we always can find a number $\eta \in \mathbb{F}$ with $|\eta - \zeta_1| < \varepsilon/2$ so that $|p(\eta)| = |p(\eta) - p(\zeta_1)| < \delta/2$. Fix such $\eta \in \mathbb{F}$ and let $\alpha = p(\eta) \in \mathbb{F}$. Then η is a root of $\tilde{P}_\alpha(0, y)$. Since $|\eta_1 - \zeta_1| < \varepsilon/2$, we have $|\eta_1 - \eta| < \varepsilon$. By $(*)$ we know $\eta_1 = \eta \in \mathbb{F}$. The lemma follows by setting $P(z, y)$ to be $P_\alpha(z, y)$. $\square$

Example 7. *The irreducible polynomial $p(y) = y^3 - 5y^2 + 3y + 2 \in \mathbb{Q}[y]$ has three real roots with approximate values $-.39138238063090084510$, 1.2271344421706896320, 4.1642479384602112131, respectively. Consider the polynomial*

$$P(z, y) = p((1 - z)y) - p(4)(1 - z) \in \mathbb{Q}[z, y].$$

This polynomial was found by the construction described in the proof, using the initial approximation 4. The equation $P(z, y) = 0$ has a solution

$$f(z) = 4 + \tfrac{46}{11}z + \tfrac{5538}{1331}z^2 + \tfrac{670794}{161051}z^3 + \tfrac{81144794}{19487171}z^4 + \tfrac{9819245130}{2357947691}z^5 + \cdots \in \mathbb{Q}[[z]],$$

the coefficients of which converge to the third root of $p(y)$. Note, for example, that the distance of the coefficient of z^4 to the root is already less than 10^{-4}. The other two roots of $P(z, y)$ are

$$\tfrac{1}{2}(1 - \sqrt{5}) + \tfrac{1}{110}(45 - 41\sqrt{5})z + \tfrac{1}{66550}(27925 - 24377\sqrt{5})z^2 + \cdots \in \bar{\mathbb{Q}}[[z]],$$
$$\tfrac{1}{2}(1 + \sqrt{5}) + \tfrac{1}{110}(45 + 41\sqrt{5})z + \tfrac{1}{66550}(27925 + 24377\sqrt{5})z^2 + \cdots \in \bar{\mathbb{Q}}[[z]].$$

Their coefficient sequences converge to the two other roots of $p(y)$, but do not belong to $\mathbb{Q}$.

The following theorem clarifies the converse direction for algebraic sequences. It turns out that every element in $\mathcal{A}_{\mathbb{F}}$ is algebraic over $\mathbb{F}$. Consequently, $\mathcal{A}_{\mathbb{F}}$ is a field.

Theorem 8. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$.*

1. *If $\mathbb{F} \subseteq \mathbb{R}$, then $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}} \cap \mathbb{R}$.*
2. *If $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$, then $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}}$.*

Proof. 1. Let $\xi \in \bar{\mathbb{F}} \cap \mathbb{R}$. Then there exists an irreducible polynomial $p(y) \in \mathbb{F}[y]$ such that $p(\xi) = 0$. By Lemma 6, ξ is equal to a limit of an algebraic sequence over $\mathbb{F}$, which implies that $\xi \in \mathcal{A}_{\mathbb{F}}$.

To show the converse inclusion, we let $\xi \in \mathcal{A}_{\mathbb{F}}$. When $\xi = 0$, there is nothing to show. Assume that $\xi \neq 0$. Then there is an algebraic sequence $(a_n)_{n=0}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = \xi$. Since $\xi \neq 0$, we have $a_n \sim \xi$ $(n \to \infty)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Clearly $f(z)$ is an algebraic function over $\mathbb{F}$. By part 2 of Theorem 5, $f(z) \sim \xi/(1 - z)$ $(z \to 1^-)$. Since $f(z)$ is algebraic, there exists a positive integer s such that $f(z)$ admits a Puiseux expansion

$$f(z) = \frac{\xi}{1 - z} + \sum_{n=-s+1}^{\infty} b_n (1 - z)^{n/s} \quad \text{with } b_n \in \mathbb{C} \text{ for all } n.$$

Setting $g(z) = f(z)(1 - z)$ establishes that

$$g(z) = \xi + \sum_{n=-s+1}^{\infty} b_n (1 - z)^{n/s+1}.$$

Note that $n/s + 1 > 0$, so $g(z)$ is finite at $z = 1$. Sending z to one gives $g(1) = \xi$. By closure properties, $g(z)$ is again an algebraic function over $\mathbb{F} \subseteq \mathbb{R}$. Thus $\xi = g(1) \in \bar{\mathbb{F}} \cap \mathbb{R}$.

2. By Lemma 6 and a similar argument as above, we have $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}}$.

□

If we were to consider the class $\mathcal{C}_{\mathbb{F}}$ of limits of convergent sequences in $\mathbb{F}$ satisfying linear recurrence equations with constant coefficients over $\mathbb{F}$, sometimes called C-finite sequences, then an argument analogous to the above proof would imply that $\mathcal{C}_{\mathbb{F}} \subseteq \mathbb{F}$, because the power series corresponding to such sequences are rational functions, and the values of rational functions over $\mathbb{F}$ at points in $\mathbb{F}$ evidently gives values in $\mathbb{F}$. The converse direction $\mathbb{F} \subseteq \mathcal{C}_{\mathbb{F}}$ is trivial, so we have $\mathcal{C}_{\mathbb{F}} = \mathbb{F}$.

Corollary 9. *If $\mathbb{F} \subseteq \mathbb{R}$, then $\bar{\mathbb{F}} = \mathcal{A}_{\mathbb{F}(i)} = \mathcal{A}_{\mathbb{F}}[i] = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}}$, where i is the imaginary unit.*

Proof. Since $\mathcal{A}_{\mathbb{F}}$ is a ring and $i^2 = -1 \in \mathbb{F} \subseteq \mathcal{A}_{\mathbb{F}}$, we have $\mathcal{A}_{\mathbb{F}}[i] = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}}$. Since $i \in \bar{\mathbb{F}}$ and $\mathbb{F} \subseteq \mathbb{R}$, the field $\bar{\mathbb{F}}$ is closed under complex conjugation and then

$$\bar{\mathbb{F}} = (\bar{\mathbb{F}} \cap \mathbb{R}) + i(\bar{\mathbb{F}} \cap \mathbb{R}) = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}},$$

by part 1 of Theorem 8. It follows from part 2 of Theorem 8 that $\mathcal{A}_{\mathbb{F}(i)} = \overline{\mathbb{F}(i)}$. Since $\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{A}_{\mathbb{F}(i)}$ and $i \in \mathcal{A}_{\mathbb{F}(i)}$,

$$\bar{\mathbb{F}} = \mathcal{A}_{\mathbb{F}} + i\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{A}_{\mathbb{F}(i)} = \overline{\mathbb{F}(i)} = \bar{\mathbb{F}},$$

The assertion holds. □

The following lemma says that every element in $\bar{\mathbb{F}}$ can be written as the value at one of an analytic algebraic function vanishing at zero, provided that $\mathbb{F}$ is dense in $\mathbb{C}$. This will be used in the next section to restrict the evaluation domain.

Lemma 10. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ with $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$. Let $p(y) \in \mathbb{F}[y]$ be an irreducible polynomial of degree d. Assume that $\zeta_1, \ldots, \zeta_d$ are all the (distinct) roots of p in $\bar{\mathbb{F}}$. Then there is a polynomial $P(z, y) \in \mathbb{F}[z, y]$ of degree d in y admitting d distinct roots $g_1(z) \in \mathbb{F}[[z]]$ and $g_2(z), \ldots, g_d(z) \in \bar{\mathbb{F}}[[z]]$ such that all $g_j(z)$ are analytic in the disk $|z| \leq 1$ with $g_j(0) = 0$ and, after reordering (if necessary), $g_j(1) = \zeta_j$.*

Proof. If $d = 1$ then $p(y) = y - \zeta_1$ with $\zeta_1 \in \mathbb{F}$. Letting $P(z, y) = y - \zeta_1 z$ yields the assertion. Otherwise $d > 1$ and all roots $\zeta_1, \ldots, \zeta_d$ are nonzero.

By Lemma 6, there exists a polynomial $\tilde{P}(z, y)$ in $\mathbb{F}[z, y]$ of degree d in y admitting d distinct roots $f_1(z) \in \mathbb{F}[[z]]$ and $f_2(z), \ldots, f_d(z) \in \bar{\mathbb{F}}[[z]]$ such that each $f_j(z)$ is analytic in the disk $|z| \leq 1$ except for a simple pole at $z = 1$ and, after reordering (if necessary),

$$\lim_{n\to\infty} [z^n] f_j(z) = \zeta_j, \quad j = 1, \ldots d.$$

Hence, together with part 2 of Theorem 5, each $f_j(z)$ admits an expansion at $z = 1$ of the form

$$f_j(z) \sim \frac{\zeta_j}{1 - z} \quad (z \to 1^-).$$

For each j set $g_j(z) = f_j(z)z(1 - z)$. Then $g_1(z) \in \mathbb{F}[[z]]$, $g_2(z), \ldots, g_d(z) \in \bar{\mathbb{F}}[[z]]$ and they are distinct from each other. Moreover, each $g_j(z)$ is analytic in the disk $|z| \leq 1$ with $g_j(0) = 0$ and $g_j(1) = \zeta_j$. By closure properties, all $g_j(z)$ are again algebraic functions. Define

$$P(z, y) = \prod_{j=1}^{d} (y - g_j(z)) = \prod_{j=1}^{d} (y - f_j(z)z(1 - z)) \in \overline{\mathbb{F}(z)}[y].$$

Since the coefficients of $P(z, y)$ w.r.t. y are symmetric in the conjugates $f_1(z), \ldots, f_d(z)$, they all belong to the field $\mathbb{F}(z)$. Multiplying P by a suitable polynomial in $\mathbb{F}[z]$ gives a desired polynomial in $\mathbb{F}[z, y]$. The lemma follows. □

4 D-finite Numbers

Let us now return to the study of D-finite numbers. Let R be a subring of $\mathbb{C}$ and $\mathbb{F}$ be a subfield of $\mathbb{C}$. Recall that by Definition 1, the elements of $\mathcal{D}_{R,\mathbb{F}}$ are exactly limits of convergent sequences in $R^{\mathbb{N}}$ which are P-recursive over $\mathbb{F}$. Some facts about P-recursive sequences translate directly into facts about $\mathcal{D}_{R,\mathbb{F}}$.

Proposition 11.

1. *$R \subseteq \mathcal{D}_{R,\mathbb{F}}$ and $\mathcal{A}_{\mathbb{F}} \subseteq \mathcal{D}_{\mathbb{F}}$.*
2. *If $R_1 \subseteq R_2$ then $\mathcal{D}_{R_1,\mathbb{F}} \subseteq \mathcal{D}_{R_2,\mathbb{F}}$, and if $\mathbb{F} \subseteq \mathbb{E}$ then $\mathcal{D}_{R,\mathbb{F}} \subseteq \mathcal{D}_{R,\mathbb{E}}$.*
3. *$\mathcal{D}_{R,\mathbb{F}}$ is a subring of $\mathbb{C}$. Moreover, if R is an $\mathbb{F}$-algebra then so is $\mathcal{D}_{R,\mathbb{F}}$.*
4. *If $\mathbb{E}$ is an algebraic extension field of $\mathbb{F}$, then $\mathcal{D}_{R,\mathbb{F}} = \mathcal{D}_{R,\mathbb{E}}$.*
5. *If $R \subseteq \mathbb{F}$, then $\mathcal{D}_{R,\mathbb{F}} = \mathcal{D}_{R,\mathrm{Quot}(R)}$.*
6. *If R and $\mathbb{F}$ are closed under complex conjugation, then so is $\mathcal{D}_{R,\mathbb{F}}$. In this case, we have $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$. Moreover, if the imaginary unit $i \in \mathcal{D}_{R,\mathbb{F}}$ then $\mathcal{D}_{R,\mathbb{F}} = \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}} + i\mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$.*

Proof.

1. The first inclusion is clear because every element of R is the limit of a constant sequence, and every constant sequence is P-recursive. The second inclusion follows from the fact that algebraic functions are D-finite, and the coefficient sequences of D-finite functions are P-recursive.
2. Clear.
3. Follows directly from the corresponding closure properties for P-recursive sequences.
4. Follows directly from part 1 of Lemma 4.
5. Follows directly from part 2 of Lemma 4.
6. For any convergent sequence $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$, we have
$$\mathrm{Re}\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} \mathrm{Re}(a_n),\ \mathrm{Im}\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} \mathrm{Im}(a_n),$$
and thus $\overline{\lim_{n\to\infty} a_n} = \lim_{n\to\infty} \bar{a}_n$. Hence the first assertion follows by $(\bar{a}_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ and part 3 of Lemma 4. Since R is closed under complex conjugation, $(\mathrm{Re}(a_n))_{n=0}^{\infty} \in (R\cap\mathbb{R})^{\mathbb{N}}$. Then the inclusion $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} \subseteq \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$ can be shown similarly as the first assertion. The converse direction holds by part 2. Therefore $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$. Moreover, if i belongs to $\mathcal{D}_{R,\mathbb{F}}$, then $\mathcal{D}_{R\cap\mathbb{R},\mathbb{F}} + i\mathcal{D}_{R\cap\mathbb{R},\mathbb{F}} \subseteq \mathcal{D}_{R,\mathbb{F}}$ since $\mathcal{D}_{R\cap\mathbb{R},\mathbb{F}} \subseteq \mathcal{D}_{R,\mathbb{F}}$. To show the converse inclusion, let $\xi \in \mathcal{D}_{R,\mathbb{F}}$. Then $\bar{\xi} \in \mathcal{D}_{R,\mathbb{F}}$ by the first assertion. Since $i \in \mathcal{D}_{R,\mathbb{F}}$ and R is closed under complex conjugation, $\mathrm{Re}(\xi), \mathrm{Im}(\xi)$ both belong to $\mathcal{D}_{R,\mathbb{F}} \cap \mathbb{R} = \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$ by the second assertion. Therefore $\xi = \mathrm{Re}(\xi) + i\,\mathrm{Im}(\xi) \in \mathcal{D}_{R\cap\mathbb{R},\mathbb{F}} + i\mathcal{D}_{R\cap\mathbb{R},\mathbb{F}}$.

∎

Example 12.

1. *We have* $\mathcal{D}_{\mathbb{Q}(\sqrt{2}),\mathbb{Q}(\pi,\sqrt{2})} = \mathcal{D}_{\mathbb{Q}(\sqrt{2}),\mathbb{Q}(\sqrt{2})} = \mathcal{D}_{\mathbb{Q}(\sqrt{2}),\mathbb{Q}}$. *The first identity holds by part 5, the second by part 4 of the proposition.*

2. *We have* $\mathcal{D}_{\bar{\mathbb{Q}},\mathbb{Q}} = \mathcal{D}_{\bar{\mathbb{Q}},\mathbb{R}}$. *The inclusion "*$\subseteq$*" is clear by part 2. For the inclusion "*$\supseteq$*", let* $\xi \in \mathcal{D}_{\bar{\mathbb{Q}},\mathbb{R}}$. *Then* $\xi = a + ib$ *for some* $a, b \in \mathbb{R}$, *and there exists a sequence* $(a_n + ib_n)_{n=0}^{\infty}$ *in* $\bar{\mathbb{Q}}^{\mathbb{N}}$ *and an operator* $L \in \mathbb{R}[n]\langle S_n\rangle$ *such that* $L \cdot (a_n + ib_n) = 0$ *and* $\lim_{n\to\infty}(a_n + ib_n) = a + ib$. *Since the coefficients of* L *are real, we then have* $L \cdot a_n = 0$ *and* $L \cdot b_n = 0$. *Furthermore,* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Therefore,*
$$a, b \in \mathcal{D}_{\bar{\mathbb{Q}}\cap\mathbb{R},\mathbb{R}} \overset{\textit{part 5}}{=} \mathcal{D}_{\bar{\mathbb{Q}}\cap\mathbb{R},\bar{\mathbb{Q}}\cap\mathbb{R}} \overset{\textit{part 4}}{=} \mathcal{D}_{\bar{\mathbb{Q}}\cap\mathbb{R},\mathbb{Q}}.$$
Hence $a + ib \in \mathcal{D}_{\bar{\mathbb{Q}}\cap\mathbb{R},\mathbb{Q}} + i\mathcal{D}_{\bar{\mathbb{Q}}\cap\mathbb{R},\mathbb{Q}} \overset{\textit{part 6}}{=} \mathcal{D}_{\bar{\mathbb{Q}},\mathbb{Q}}$, *as claimed.*

The results for C-finite and algebraic cases motivate the following theorem, which says that every D-finite number is essentially the (left) limiting value at one of a D-finite function.

Theorem 13. *Let* R *be a subring of* $\mathbb{C}$ *and let* $\mathbb{F}$ *be a subfield of* $\mathbb{C}$. *Then for every* $\xi \in \mathcal{D}_{R,\mathbb{F}}$, *there exists* $g(z) \in R[[z]]$ *D-finite over* $\mathbb{F}$ *such that* $\xi = \lim_{z\to 1^-} g(z)$.

Proof. The statement is clear when $\xi = 0$. Assume that ξ is nonzero. Then there exists a sequence $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ P-recursive over $\mathbb{F}$ such that $\lim_{n\to\infty} a_n = \xi$. Since $\xi \neq 0$, we have $a_n \sim \xi$ $(n \to \infty)$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. By part 2 of Theorem 5, $f(z) \sim \xi/(1-z)$ as $z \to 1^-$, which, by definition, implies that
$$\lim_{z\to 1^-} \frac{f(z)}{\xi/(1-z)} = \lim_{z\to 1^-} \frac{(1-z)f(z)}{\xi} = 1.$$
Letting $g(z) = f(z)(1-z)$ gives $\lim_{z\to 1^-} g(z)/\xi = 1$, and then $\lim_{z\to 1^-} g(z) = \xi$ since $\xi \neq 0$. The assertion follows by noticing that $g(z) \in R[[z]]$ is D-finite over $\mathbb{F}$ due to closure properties. $\square$

Example 14. *We have* $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \lim_{z\to 1^-} \mathrm{Li}_3(z) = \mathrm{Li}_3(1)$, *where* $\mathrm{Li}_3(z) = \sum_{n=1}^{\infty} \frac{1}{n^3} z^n$ *is the polylogarithm function. Note that* $\mathrm{Li}_3(z) \in \mathbb{Q}[[z]]$ *is D-finite over* $\mathbb{Q}$ *and finite at* $z = 1$.

Note that the above theorem implies that D-finite numbers are computable when the ring R and the field $\mathbb{F}$ consist of computable numbers. This allows the construction of (artificial) numbers that are not D-finite.

We next turn to some sort of converse of Theorem 13. To this end, we need to develop several lemmas. First note that we may assume without loss of generality that $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$, because for any $\mathbb{F}$ we will always have $\mathbb{F}(i) \setminus \mathbb{R} \neq \emptyset$ and, by part 4 of Prop. 11, $\mathcal{D}_{R,\mathbb{F}} = \mathcal{D}_{R,\mathbb{F}(i)}$, so we can always replace $\mathbb{F}$ by $\mathbb{F}(i)$. Let us thus assume $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ for the remainder of this section.

The first lemma says that the value of a D-finite function at any non-singular point in $\bar{\mathbb{F}}$ can be represented by the value of another D-finite function at one.

Lemma 15. *Let* $\mathbb{F}$ *be a subfield of* $\mathbb{C}$ *with* $\mathbb{F}\setminus\mathbb{R} \neq \emptyset$ *and* R *be a subring of* $\mathbb{C}$ *containing* $\mathbb{F}$. *Assume that* $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ *is analytic at zero and annihilated by a nonzero operator* $L \in \mathbb{F}[z]\langle D_z\rangle$ *with zero being an ordinary point. Then for any non-singular point* $\zeta \in \bar{\mathbb{F}}$ *of* L, *there exists* $h(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ *and* $M \in \mathbb{F}[z]\langle D_z\rangle$ *nonzero with zero and one being ordinary points such that* $M \cdot h(z) = 0$ *and* $f(\zeta) = h(1)$.

Proof. Let $\zeta_1 \in \bar{\mathbb{F}}$ be a non-singular point of L. Then there exists an irreducible polynomial $p(z) \in \mathbb{F}[z]$ of degree d such that $p(\zeta_1) = 0$. Let $\zeta_2, \ldots, \zeta_d$ be all other roots of p in $\bar{\mathbb{F}}$. By Lemma 10, there exists a polynomial $P(z,y) \in \mathbb{F}[z,y]$ of degree d in y admitting d distinct roots $g_1(z) \in \mathbb{F}[[z]]$ and $g_2(z), \ldots, g_d(z) \in \bar{\mathbb{F}}[[z]]$ such that all $g_j(z)$ are analytic in the disk $|z| \le 1$ with $g_j(0) = 0$ and $g_j(1) = \zeta_j$. In particular, all $g_j(z)$ are analytic in a neighborhood of zero including the point $z = 1$.

Since $g_1(1) = \zeta_1$ is not a singularity of L by assumption, none of $g_j(1) = \zeta_j$ is a singularity of L. In fact, suppose otherwise that for some $2 \le \ell \le d$, the point $g_\ell(1) = \zeta_\ell$ is a root of $\operatorname{lc}(L)$. Since $\operatorname{lc}(L) \in \mathbb{F}[z]$ and p is the minimal polynomial of ζ_ℓ over $\mathbb{F}$, we know that $p(z)$ divides $\operatorname{lc}(L)$ over $\mathbb{F}$. Thus ζ_1 is also a root of $\operatorname{lc}(L)$, a contradiction.

By assumption, zero is an ordinary point of L. Note that $g_j(0) = 0$ for all j. It follows from Theorem 3 that there exists a nonzero operator $M \in \mathbb{F}[z]\langle D_z\rangle$ with $M \cdot (f \circ g_1) = 0$ which does not have zero or one among its singularities. By part 1 of Proposition 11, $\mathbb{F} \subseteq R \subseteq \mathcal{D}_{R,\mathbb{F}}$. Since $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ and $g_1(z) \in \mathbb{F}[[z]]$ with $g_1(0) = 0$, we have $f(g_1(z)) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$. Setting $h(z) = f(g_1(z))$ completes the proof. □

With the above lemma, it suffices to consider the case when the evaluation point is in $R \cap \mathbb{F}$. This is exactly what the next two lemmas are concerned about.

Lemma 16. *Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R[[z]]$ is D-finite over $\mathbb{F}$ and convergent in some neighborhood of zero. Let $\zeta \in R \cap \mathbb{F}$ be in the disk of convergence. Then $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$.*

Proof. For any $k \in \mathbb{N}$, it is well-known that $f^{(k)}(z) \in R[[z]]$ is also D-finite and has the same radius of convergence at zero as $f(z)$. Note that since $f(z)$ is D-finite over $\mathbb{F}$, so is $f^{(k)}(z)$. Thus to prove the lemma, it suffices to show the case when $k = 0$, i.e., $f(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$.

Since $f(z)$ is D-finite over $\mathbb{F}$, the coefficient sequence $(a_n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{F}$. Note that $\zeta \in R \cap \mathbb{F}$ is in the disk of convergence of $f(z)$ at zero, so

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \lim_{n\to\infty} \sum_{\ell=0}^{n} a_\ell \zeta^\ell.$$

Since $(\zeta^n)_{n=0}^{\infty}$ is P-recursive over $\mathbb{F}$, the assertion follows by noticing that $(\sum_{\ell=0}^{n} a_\ell \zeta^\ell)_{n=0}^{\infty} \in R^{\mathbb{N}}$ is P-recursive over $\mathbb{F}$ due to closure properties. □

Example 17.

1. *Since* $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \in \mathbb{Q}[[z]]$ *is D-finite over $\mathbb{Q}$, and converges everywhere, we get from the lemma that the numbers* $\mathrm{e}, 1/\mathrm{e}, \sqrt{\mathrm{e}}$ *belong to $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}$. More precisely, since we are currently only considering non-real fields $\mathbb{F}$, we could say that* $\exp(z)$ *is D-finite over $\bar{\mathbb{Q}}$, therefore* $\mathrm{e}, 1/\mathrm{e}, \sqrt{\mathrm{e}} \in \mathcal{D}_{\mathbb{Q},\bar{\mathbb{Q}}}$*, but by Proposition 11, $\mathcal{D}_{\mathbb{Q},\bar{\mathbb{Q}}} = \mathcal{D}_{\mathbb{Q},\mathbb{Q}}$.*

2. *All (finite) values of G-function at algebraic numbers belong to $\mathcal{D}_{\mathbb{Q}(i)}$, as remarked in the introduction. Indeed, [4, Theorem 1] tells us that any complex number ξ that appears as the value of a G-function at some algebraic number has real and imaginary parts both of the form $f(1)$ for some G-function $f(z)$ with rational coefficients and whose radius of convergence is greater than one. Together with the above lemma, we readily see that such ξ belong to $\mathcal{D}_{\mathbb{Q},\mathbb{Q}(i)} + i\mathcal{D}_{\mathbb{Q},\mathbb{Q}(i)}$, which is actually equal to $\mathcal{D}_{\mathbb{Q}(i)}$ by part 6 of Proposition 11.*

Lemma 18. *Let R be a subring of $\mathbb{C}$ containing $\mathbb{F}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ be an analytic function at zero. Assume that there exists a nonzero operator $L \in \mathbb{F}[z]\langle D_z\rangle$ with zero being an ordinary point such that $L \cdot f(z) = 0$. Let $r > 0$ be the smallest modulus of roots of $\operatorname{lc}(L)$ and let $\zeta \in \mathbb{F}$ with $|\zeta| < r$. Then $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$.*

Proof. Let ρ be the order of L. Since zero is an ordinary point of L, there exist P-recursive sequences $(c_n^{(0)})_{n=0}^{\infty}, \ldots, (c_n^{(\rho-1)})_{n=0}^{\infty}$ in $\mathbb{F}^{\mathbb{N}} \subseteq R^{\mathbb{N}}$ with $c_j^{(m)}$ equal to the Kronecker delta δ_{mj} for $m, j = 0, \ldots, \rho - 1$, so that the set $\{\sum_{n=0}^{\infty} c_n^{(m)} z^n\}_{m=0}^{\rho-1}$ forms a basis of the solution space of L near zero. Since $L \cdot f(z) = 0$ and $f(z)$ is analytic at zero,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 \sum_{n=0}^{\infty} c_n^{(0)} z^n + \cdots + a_{\rho-1} \sum_{n=0}^{\infty} c_n^{(\rho-1)} z^n \tag{1}$$

near zero. Note that the singularities of solutions of L can only be roots of $\operatorname{lc}(L)$. Hence $f(z)$ as well as $\sum_{n=0}^{\infty} c_n^{(m)} z^n$ for $m = 0, \ldots, \rho-1$ are convergent in the disk $|z| < r$. It follows from $|\zeta| < r$ and Lemma 16 that the set $\{\sum_{n=0}^{\infty} c_n^{(m)} \zeta^n\}_{m=0}^{\rho-1}$ belongs to $\mathcal{D}_{R,\mathbb{F}}$. Since $a_0, \ldots, a_{\rho-1} \in \mathcal{D}_{R,\mathbb{F}}$, letting $z = \zeta$ in Eq. (1) yields that $f(\zeta)$ is D-finite by closure properties. Differentiating Eq. (1), we find by Lemma 16 that for $k > 0$, the derivative $f^{(k)}(\zeta)$ also belongs to $\mathcal{D}_{R,\mathbb{F}}$. □

Example 19.

1. *We know from Proposition 11 that $\sqrt{2} \in \mathcal{D}_{\mathbb{Q}}$. The series*

$$(z+1)^{\sqrt{2}} = 1 + \sqrt{2}z + (1 - \frac{1}{\sqrt{2}})z^2 + \cdots \in \mathbb{Q}(\sqrt{2})[[z]] \subseteq \mathcal{D}_{\mathbb{Q}}[[z]]$$

is D-finite over $\mathbb{Q}$, an annihilating operator is $(z+1)^2 D_z^2 + (z+1)D_z - 2$. Here we have the radius $r = 1$. Taking $\zeta = \sqrt{2} - 1$, the lemma implies $\sqrt{2}^{\sqrt{2}} \in \mathcal{D}_{\mathbb{Q}}$.

2. *Observe that the lemma refers to the singularities of the operator rather than to the singularities of the particular solution at hand. For example, it does not imply that $J_1(1) \in \mathcal{D}_{\mathbb{Q},\mathbb{Q}}$, where $J_1(z)$ is the first Bessel function, because its annihilating operator is $z^2 D_z^2 + zD_z + (z^2 - 1)$, which has a singularity at zero. It is not sufficient that the particular solution $J_1(z) \in \mathbb{Q}[[z]]$ is analytic at zero. Of course, in this particular example we see from the series representation $J_1(1) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1/4)^n}{(n+1)n!^2}$ that the value belongs to $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}$.*

3. *The hypergeometric function $f(z) := {}_2F_1(\frac{1}{3}, \frac{1}{2}, 1, z + \frac{1}{2})$ can be viewed as an element of $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}[[z]]$:*

$$f(z) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2^n n!} \sum_{k=n}^{\infty} \frac{(1/2)_k (1/3)_k}{k!(k-n)!} \left(\frac{1}{2}\right)^k\right]}_{\in \mathcal{D}_{\mathbb{Q},\mathbb{Q}}} z^n.$$

The function f is annihilated by the operator

$$L = 3(2z-1)(2z+1)D_z^2 + (22z-1)D_z + 2.$$

This operator has a singularity at $z = 1/2$, and there is no annihilating operator of f which does not have a singularity there. Although $f(1/2) = \frac{\Gamma(1/6)}{\Gamma(1/2)\Gamma(2/3)}$ is a finite and specific value, the lemma does not imply that this value is a D-finite number.

Theorem 20. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ with $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ and let R be a subring of $\mathbb{C}$ containing $\mathbb{F}$. Assume that $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z]]$ is analytic at zero and there exists a nonzero operator $L \in \mathbb{F}[z]\langle D_z\rangle$ with zero being an ordinary point so that $L \cdot f(z) = 0$. Further assume that $\zeta \in \bar{\mathbb{F}}$ is not a singularity of L. Then $f^{(k)}(\zeta) \in \mathcal{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$.*

Proof. By Lemma 15, it suffices to show that the assertion holds for $\zeta = 1$ (or more generally $\zeta \in \mathbb{F}$). Now assume that $\zeta \in \mathbb{F}$. We apply the method of analytic continuation.

Let $\mathcal{P}$ be a simple path with a finite cover $\bigcup_{j=0}^{s} \mathcal{B}_{r_j}(\beta_j)$, where $s \in \mathbb{N}$, $\beta_0 = 0$, $\beta_s = \zeta$, $\beta_j \in \mathbb{F}$, $r_j > 0$ is the distance between β_j and the zero set of $\operatorname{lc}(L)$, and $\mathcal{B}_{r_j}(\beta_j)$ is the open ball centered at β_j and with radius r_j. Moreover, $\beta_{j+1} \in \mathcal{B}_{r_j}(\beta_j)$ for each j (as illustrated by Fig. 2). Such a path exists because $\mathbb{F}$ is dense in $\mathbb{C}$ and the zero set of $\operatorname{lc}(L)$ is finite. Since the path $\mathcal{P}$ avoids all roots of $\operatorname{lc}(L)$, the function $f(z)$ is analytic along $\mathcal{P}$. We next use induction on the index j to show that $f^{(k)}(\beta_j) \in \mathcal{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$.

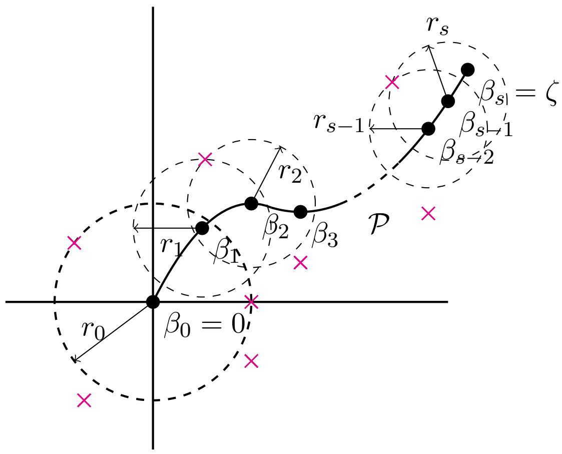

Figure 2: a simple path $\mathcal{P}$ with a finite cover $\bigcup_{j=0}^{s} \mathcal{B}_{r_j}(\beta_j)$ ($\times$ stands for the roots of $\operatorname{lc}(L)$)

It is trivial when $j = 0$ as $f^{(k)}(\beta_0) = f^{(k)}(0) \in \mathcal{D}_{R,\mathbb{F}}$ for $k \in \mathbb{N}$ by assumption. Assume now that $0 < j \leq s$ and $f^{(k)}(\beta_{j-1}) \in \mathcal{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$. We consider $f(\beta_j)$ and its derivatives.

Recall that $r_{j-1} > 0$ is the distance between β_{j-1} and the zero set of $\operatorname{lc}(L)$. Since $f(z)$ is analytic at β_{j-1}, it is representable by a convergent power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\beta_{j-1})}{n!}(z - \beta_{j-1})^n \quad \text{for all } |z - \beta_{j-1}| < r_{j-1}.$$

By the induction hypothesis, $f^{(n)}(\beta_{j-1})/n! \in \mathcal{D}_{R,\mathbb{F}}$ for all $n \in \mathbb{N}$ and thus $f(z) \in \mathcal{D}_{R,\mathbb{F}}[[z - \beta_{j-1}]]$. Let $Z = z - \beta_{j-1}$, i.e., $z = Z + \beta_{j-1}$. Define $g(Z) = f(Z + \beta_{j-1})$ and $\tilde{L}$ to be the operator obtained by replacing z in L by $Z + \beta_j$. Since $\beta_{j-1} \in \mathbb{F} \subseteq \mathcal{D}_{R,\mathbb{F}}$ and $D_z = D_Z$, we have $g(Z) \in \mathcal{D}_{R,\mathbb{F}}[[Z]]$ and $\tilde{L} \in \mathbb{F}[Z]\langle D_Z\rangle$. Note that $L \cdot f(z) = 0$ and β_{j-1} is an ordinary point of L as $r_{j-1} > 0$. It follows that $\tilde{L} \cdot g(Z) = 0$ and zero is an ordinary point of $\tilde{L}$. Moreover, we see that r_{j-1} is now the smallest modulus of roots of $\operatorname{lc}(\tilde{L})$. Since $|\beta_j - \beta_{j-1}| < r_{j-1}$, applying Lemma 18 to $g(Z)$ yields $f^{(k)}(\beta_j) = g^{(k)}(\beta_j - \beta_{j-1}) \in \mathcal{D}_{R,\mathbb{F}}$ for $k \in \mathbb{N}$. Thus the assertion holds for $j = s$. The theorem follows. $\square$

Example 21. *By the above theorem,* $\exp(\sqrt{2})$ *and* $\log(1+\sqrt{3})$ *both belong to* $\mathcal{D}_{\mathbb{Q}}$*. We also have* $e^{\pi} \in \mathcal{D}_{\mathbb{Q}}$*. This is because* $e^{\pi} = (-1)^{-i}$*, with* i *the imaginary unit, is equal to the value of the*

D-finite function $(z+1)^{-i} \in \mathbb{Q}(i)[[z]]$ *at* $z=-2$ *(outside the radius of convergence; analytically continued in consistency with the usual branch cut conventions) and then* $\mathrm{e}^{\pi} \in \mathcal{D}_{\mathbb{Q}(i)} \cap \mathbb{R} = \mathcal{D}_{\mathbb{Q}}$.

5 Open Questions

We have introduced the notion of D-finite numbers and made some first steps towards understanding their nature. We believe that, similarly as for D-finite functions, the class is interesting because it has good mathematical and computational properties and because it contains many special numbers that are of independent interest. We conclude this paper with some possible directions of future research.

Evaluation at singularities. While every singularity of a D-finite function must also be a singularity of its annihilating operator, the converse is in general not true. We have seen above that evaluating a D-finite function at a point which is not a singularity of its annihilating operator yields a D-finite number. It would be natural to wonder about the values of a D-finite function at singularities of its annihilating operator, including those at which the given function is not analytic but its evaluation is finite. Also, we always consider zero as an ordinary point of the annihilating operator. If this is not the case, the method used in the paper fails, as pointed out by part 2 of Example 19.

Quotients of D-finite numbers. The set of algebraic numbers forms a field, but we do not have a similar result for D-finite numbers. It is known that the set of D-finite functions does not form a field. Instead, Harris and Sibuya [10] showed that a D-finite function f admits a D-finite multiplicative inverse if and only if f'/f is algebraic. This explains for example why both e and $1/\mathrm{e}$ are D-finite, but it does not explain why both π and $1/\pi$ are D-finite. It would be interesting to know more precisely under which circumstances the multiplicative inverse of a D-finite number is D-finite. Is $1/\log(2)$ a D-finite number? Are there choices of R and $\mathbb{F}$ for which $\mathcal{D}_{R,\mathbb{F}}$ is a field?

Roots of D-finite functions. A similar pending analogy concerns compositional inverses. We know that if f is an algebraic function, then so is its compositional inverse f^{-1}. The analogous statement for D-finite functions is not true. Nevertheless, it could still be true that the values of compositional inverses of D-finite functions are D-finite numbers, although this seems somewhat unlikely. A particularly interesting special case is the question whether (or under which circumstances) the roots of a D-finite function are D-finite numbers.

Evaluation at D-finite number arguments. We see that the class $\mathcal{C}_{\mathbb{F}}$ of limits of convergent C-finite sequences is the same as the values of rational functions at points in $\mathbb{F}$, namely the field $\mathbb{F}$. Similarly, the class $\mathcal{A}_{\mathbb{F}}$ of limits of convergent algebraic sequences essentially consists of the values of algebraic functions at points in $\bar{\mathbb{F}}$. Continuing this pattern, is the value of a D-finite function at a D-finite number again a D-finite number? If so, this would imply that also numbers like $\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\mathrm{e}}}}$ are D-finite. Since $1/(1-z)$ is a D-finite function, it would also imply that D-finite numbers form a field.

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