A q-Analogue of the Modified Abramov-Petkovšek Reduction

Dedicated to Professor Sergei A. Abramov on the occasion of his 70th birthday

Hao Du, Hui Huang and Ziming Li

Abstract We present an additive decomposition algorithm for q-hypergeometric terms. It decomposes a given term T as the sum of two terms, in which the former is q-summable and the latter is minimal in some technical sense. Moreover, the latter is zero if and only if T is q-summable. Although our additive decomposition is a q-analogue of the modified Abramov-Petkovšek reduction for usual hypergeometric terms, they differ in some subtle details. For instance, we need to reduce Laurent polynomials instead of polynomials in the q-case. The experimental results illustrate that the additive decomposition is more efficient than q-Gosper's algorithm for determining q-summability when some q-dispersion concerning the input term becomes large. Moreover, the additive decomposition may serve as a starting point to develop a reduction-based creative-telescoping method for q-hypergeometric terms.

1 Introduction

q-Hypergeometric terms are basic objects in q-analysis. An important question concerning q-hypergeometric terms is to decide whether such a term is the q-difference of another term of the same kind. This question is referred to as the *q-summability problem*, which can be solved by a direct q-analogue of Gosper's algorithm developed by Koornwinder in [12], or an algebraically motivated q-analogue by Paule

---

Hao Du
KLMM, AMSS, Chinese Academy of Sciences, Beijing, China. e-mail: duhao@amss.ac.cn

Hui Huang
Institute for Algebra, Johannes Kepler University, Linz, Austria. e-mail: huanghui@amss.ac.cn

Ziming Li
KLMM, AMSS, Chinese Academy of Sciences, Beijing, China;
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China. e-mail: zmli@mmrc.iss.ac.cn

H. Du was supported by the NSFC grants 11501552. H. Huang was funded by the Austrian Science Fund (FWF) under grants Y464-N18 and W1214-N15 (project part 13).

and Riese in [14]. Both analogues need to compute a polynomial solution of some auxiliary linear q-recurrence equation of first order.

An alternative approach to dealing with q-summability problem is to decompose a given q-hypergeometric term as the sum of two terms of the same kind such that the former is q-summable and the latter is minimal in some technical sense. Moreover, the latter is zero if and only if the given term is q-summable. How to compute such a decomposition is referred to as the *q-additive decomposition problem*.

A rational function over a field of constants can be viewed as a usual hypergeometric or a q-hypergeometric term. For the usual shift case, Abramov in [1] developed an algorithm to decompose a rational function into a summable rational function and a nonsummable one whose denominator is of least possible degree. Moreover, a rational function is summable if and only if the nonsummable one in its additive decomposition is equal to zero. Abramov's algorithm can be easily adapted for solving the additive decomposition problem in the q-shift case. Both Abramov's algorithm and its q-analogue do not require computing polynomial solutions of any auxiliary (q-)recurrence equation. Schneider in [15] worked out a general approach to decomposing a rational function over a nonconstant difference field under the assumption that some parametric linear recurrence equation of first order is solvable in the difference field.

Abramov and Petkovšek developed an algorithm for computing an additive decomposition for usual hypergeometric terms in [2, 3]. We call it the Abramov-Petkovšek reduction. Their algorithm needs to compute polynomial solutions of some auxiliary recurrence equation. Part of the Abramov-Petkovšek reduction is translated to the q-case by Chen, Hou and Mu in [9] on the way to establish a criterion on the termination of the q-analogue of Zeilberger's algorithm. Chen *et al* in [6] present a modified Abramov-Petkovšek reduction for usual hypergeometric terms to avoid computing polynomial solutions of any auxiliary recurrence equation. This feature is crucial for reduction-based creative-telescoping methods.

The goal of this paper is to further develop a q-analogue of the modified Abramov-Petkovšek reduction, which provides a solution to the q-additive decomposition problem. The analogue also avoids solving any auxiliary q-recurrence equation. Similar to the modified Abramov-Petkovšek reduction, it consists of two steps, namely, shell and polynomial reductions. In the usual shift case, the shell reduction was carried out by calculating dispersions and partial fraction decomposition. When implementing its q-analogue in Maple, we observe that a combination of q-shift homogeneous factorization [13, 3] with the above two calculations yields an overall better performance. This is because the partial fraction decomposition of q-rational functions tends to be faster when their denominators split into powers of irreducible factors, which is particularly true when q is an indeterminate. So the step for shell reduction is described in terms of q-shift homogeneous factorization. Moreover, in order to obtain q-shift-free denominators, we need to allow some numerators to be Laurent polynomials, which complicates the step for polynomial reduction. Experimental results illustrate that the q-analogue of the modified Abramov-Petkovšek reduction outperforms q-Gosper's algorithm when the q-dispersions of the denominators of shells become large. Please see Section 6 for more details. Hopefully, this

q-analogue may enable us to develop a reduction-based creative-telescoping method for q-hypergeometric terms in a similar way as in [4, 6].

2 Summability and congruences

Throughout the paper, let C be a field of characteristic zero, and σ be an automorphism of $C[x]$ such that C is the subfield of constants with respect to σ. Then $\sigma(x) = \lambda x + \mu$ with $\lambda \in C \setminus \{0\}$ and $\mu \in C$, where either $\mu \neq 0$ or $\lambda \neq 1$ (cf. [11]). We call σ the usual shift operator if $(\lambda, \mu) = (1, 1)$; and call it a q-shift operator if $(\lambda, \mu) = (q, 0)$, where q is not a root of unity. The automorphism σ can be naturally extended to $C(x)$. Let Δ be the difference operator $\sigma - \mathbf{1}$ on $C(x)$, where $\mathbf{1}$ stands for the identity map from $C(x)$ to itself.

Let R be a ring extension of $C(x)$. Assume that σ can be extended to a monomorphism of R. An element $r \in R$ is called a *constant* if $\sigma(r) = r$. The subset of constants in R forms a subring, which is denoted by C_R.

An invertible element T of R is said to be *hypergeometric* with respect to σ or σ-*hypergeometric* for short if its σ-quotient $\sigma(T)/T$ belongs to $C(x)$. Every nonzero element of $C(x)$ is σ-hypergeometric. When σ is a q-shift operator, σ-hypergeometric terms are also called *q-hypergeometric terms*. All conclusions in this section are valid for general σ-hypergeometric terms.

Two σ-hypergeometric terms are said to be *similar* if their ratio belongs to $C(x)$. A σ-hypergeometric term T is said to be *summable* if there exists another σ-hypergeometric term G such that $T = \Delta(G)$. It is straightforward to verify that two σ-hypergeometric terms T and G are similar if $T = \Delta(G)$. A key idea on determining summability of a given σ-hypergeometric term T is to write $T = fH$, where f is a nonzero element of $C(x)$, and H is another σ-hypergeometric term whose σ-quotient satisfies certain properties (see [2]). With such a multiplicative decomposition at hand, we see that determining the summability of T amounts to finding a rational function g such that $fH = \Delta(gH)$. Assume that K is the σ-quotient of H. Then fH is summable if and only if $f = K\sigma(g) - g$ for some $g \in C(x)$. In other words, determining the summability of fH amounts to finding a rational solution of the first-order linear recurrence equation $K\sigma(z) - z = f$.

Let us formulate the above deduction in a different way, which will be convenient to describe various congruences in the sequel. Let K be a nonzero rational function in $C(x)$. Then $K\sigma$ is a C-linear automorphism of $C(x)$ that maps f to $K\sigma(f)$. We define a C-linear map $\Delta_K = K\sigma - \mathbf{1}$ from $C(x)$ to itself. Then, for any σ-hypergeometric term H with $\sigma(H)/H = K$, we have fH is summable if and only if $f \in \operatorname{im}(\Delta_K)$. This image is a C-linear subspace contained in $C(x)$.

Our reduction in the sequel relies on four congruences modulo the image of Δ_K. The first two congruences are given below.

Lemma 1. *Let K be a nonzero rational function in $C(x)$. Then, for every $f \in C(x)$,*

$$f \equiv K\sigma(f) \mod \operatorname{im}(\Delta_K) \quad \textit{and} \quad f \equiv (K\sigma)^{-1}(f) \mod \operatorname{im}(\Delta_K).$$

Proof. The first congruence follows immediately from the definition of Δ_K. To prove the second one, we note that $K\sigma$ is a bijection. Therefore, there exists $g \in C(x)$ such that $f = K\sigma(g)$. By the first congruence, $g \equiv K\sigma(g) \mod \operatorname{im}(\Delta_K)$. Replacing g with $(K\sigma)^{-1}(f)$ yields the second congruence. □

Corollary 1. *Let K be a nonzero rational function of $C(x)$. Then, for every $f \in C(x)$ and $m \in \mathbb{N}$, we have*

$$f \equiv \sigma^m(f) \prod_{i=0}^{m-1} \sigma^i(K) \mod \operatorname{im}(\Delta_K)$$

and

$$f \equiv \sigma^{-m}(f) \prod_{i=1}^{m} \sigma^{-i}\left(K^{-1}\right) \mod \operatorname{im}(\Delta_K).$$

Proof. By Lemma 1 and a straightforward induction, we see that

$$f \equiv (K\sigma)^m(f) \mod \operatorname{im}(\Delta_K) \quad \text{and} \quad f \equiv (K\sigma)^{-m}(f) \mod \operatorname{im}(\Delta_K).$$

The corollary follows from the definition of $K\sigma$ and its inverse. □

The two congruences in the above corollary will be called the *forward and backward congruences*, respectively.

Remark 1. The two congruences in Lemma 1 can be translated into two equalities:

$$f = \Delta_K(-f) + K\sigma(f) \quad \text{and} \quad f = \Delta_K(g) + (K\sigma)^{-1}(f),$$

where $g = (K\sigma)^{-1}(f)$. It follows that both forward and backward congruences can be translated into equalities.

The notions of shift and q-shift reduced rational functions are introduced in [2] and [9], respectively. We extend them slightly, because the next two congruences hold in both shift and q-shift cases. Let $K \in C(x)$ be a nonzero rational function with numerator u and denominator v. We say that K is *reduced with respect to* σ or σ*-reduced* for short if u and $\sigma^i(v)$ are relatively prime for all $i \in \mathbb{Z}$.

Lemma 2. *Let $K \in C(x)$ be a σ-reduced rational function with numerator u and denominator v. Then, for every $a \in C[x]$ and $n \in \mathbb{N}$, there exist two polynomials b_1 and b_2 in $C[x]$ such that*

$$\frac{a}{\prod_{i=0}^{n} \sigma^i(v)} \equiv \frac{b_1}{v} \mod \operatorname{im}(\Delta_K) \quad \textit{and} \quad \frac{a}{\prod_{j=1}^{n} \sigma^{-j}(u)} \equiv \frac{b_2}{v} \mod \operatorname{im}(\Delta_K).$$

Proof. We prove the first congruence by induction on n. Let $w_n = \prod_{i=0}^{n} \sigma^i(v)$. The congruence is trivial when $n = 0$. Assume that it holds for $n-1$. Setting $f = a/w_n$ in the second congruence in Lemma 1, we see that

$$\frac{a}{w_n} \equiv \sigma^{-1}\left(\frac{a}{w_n}\frac{v}{u}\right) = \frac{\sigma^{-1}(a)}{w_{n-1}\sigma^{-1}(u)} \mod \operatorname{im}(\Delta_K).$$

Since K is σ-reduced, $\gcd(w_{n-1}, \sigma^{-1}(u)) = 1$, there exist $e_1, e_2 \in C[x]$ such that

$$\frac{a}{w_n} \equiv \frac{\sigma^{-1}(a)}{w_{n-1}\sigma^{-1}(u)} = \frac{e_1}{w_{n-1}} + \frac{e_2}{\sigma^{-1}(u)} \mod \operatorname{im}(\Delta_K).$$

By the induction hypothesis, the first summand is congruent to b_1'/v for some b_1' in $C[x]$. Setting $f = e_2/\sigma^{-1}(u)$ in the first congruence in Lemma 1, we see that the second summand is congruent to $\sigma(e_2)/v$. Setting $b_1 = b_1' + \sigma(e_2)$ establishes the first congruence in this lemma.

To prove the second congruence, we notice that the product in the denominator equals one when $n = 0$ and then there is nothing to show in this case. For $n = 1$, we set $f = a/\sigma^{-1}(u)$ in the forward congruence in Lemma 1 to get

$$\frac{a}{\sigma^{-1}(u)} \equiv K\sigma\left(\frac{a}{\sigma^{-1}(u)}\right) = \frac{\sigma(a)}{v} \mod \operatorname{im}(\Delta_K),$$

which is exactly the second congruence with $n = 1$. The induction can be completed in a similar way as in the proof of the first congruence. □

Remark 2. In the above proof, all congruences are obtained from the congruences in Lemma 1. So they can be translated into equalities by Remark 1.

3 Kernels, shells and σ-factorizations in the q-case

From now on, we assume that σ is an automorphism of $C(x)$ such that $\sigma(x) = qx$, where q is neither zero nor any root of unity in C. According to [14], a polynomial p in $C[x]$ is said to be *q-monic* if $p(0) = 1$. Assume that p is q-monic. Then so is $\sigma^i(p)$ for all $i \in \mathbb{Z}$. If, moreover, p is irreducible, then $\sigma^i(p)$ and p are coprime for all $i \in \mathbb{Z}$ with $i \neq 0$. Let f be a nonzero rational function in $C(x)$ with denominator a and numerator b. By a factor of f, we mean a factor of either a or b. We say that f is *q-monic* if both a and b are q-monic.

For a nonzero rational function f in $C(x)$, there exist a σ-reduced rational function K and a nonzero rational function S such that

$$f = K\frac{\sigma(S)}{S}.$$

We call K a *kernel* and S the corresponding *shell* of f. They can be computed by gcd-calculations (cf. [9]).

Recall that an element of $C(x)$ is proper if its numerator has degree lower than that of the denominator, and that it is a Laurent polynomial if the denominator is a power of x. All Laurent polynomials in $C(x)$ form a subring, which is denoted by $C[x, x^{-1}]$. Every nonzero rational function can be decomposed as the sum of a Laurent polynomial and a proper rational function whose denominator is q-monic.

This decomposition enables us to deal with Laurent polynomials and proper rational functions with q-monic denominators separately.

A nonzero Laurent polynomial f can be written in the form $\sum_{i=m}^{n} c_i x^i$, where $m,n \in \mathbb{Z}$ with $m \leq n$ and $c_m, c_{m+1}, \ldots, c_n \in C$ with $c_m c_n \neq 0$. We call n the *head degree of* f and m the *tail degree of* f. They are denoted by $\operatorname{hdeg}(f)$ and $\operatorname{tdeg}(f)$, respectively. Moreover, we define $\operatorname{hdeg}(0) = -\infty$ and $\operatorname{tdeg}(0) = +\infty$. Such a convention agrees with the inequalities: for all $f, g \in C[x, x^{-1}]$,

$$\operatorname{hdeg}(f+g) \leq \max(\operatorname{hdeg}(f), \operatorname{hdeg}(g)) \quad \text{and} \quad \operatorname{tdeg}(f+g) \geq \min(\operatorname{tdeg}(f), \operatorname{tdeg}(g)).$$

Furthermore, the ring of Laurent polynomials in σ over $\mathbb{Z}$, denoted by $\mathbb{Z}[\sigma, \sigma^{-1}]$, is useful to describe a number of notions uniformly in the sequel.

Let p be a nonzero polynomial and $\alpha = \sum_{i=m}^{n} k_i \sigma^i$ be in $\mathbb{Z}[\sigma, \sigma^{-1}]$. We define

$$p^{\alpha} := \prod_{i=m}^{n} \sigma^i(p)^{k_i}.$$

Clearly, p^{α} is a polynomial if and only if α belongs to $\mathbb{N}[\sigma, \sigma^{-1}]$.

According to [11, Definition 11] and [3, Definition 1], two irreducible polynomials $a, b \in C[x]$ are said to be equivalent with respect to σ or σ-equivalent for short if $a \mid \sigma^i(b)$ for some $i \in \mathbb{Z}$. The σ-equivalence of two polynomials can be easily recognized by comparing coefficients. A rational function $f \in C(x)$ is said to be *q-shift homogeneous* if all nonconstant irreducible factors of the numerator and denominator of f belong to the same σ-equivalence class.

For any nonzero rational function $f \in C(x)$, by grouping together σ-equivalent factors of its numerator and denominator, it can be written in the form

$$f = c\, x^m \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{1}$$

where $c \in C \setminus \{0\}$, $m \in \mathbb{Z}$, $s \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}[\sigma, \sigma^{-1}]$, $p_i \in C[x]$ is nonconstant, q-monic and irreducible for $i = 1, \ldots, s$, and the p_i's are pairwise inequivalent with respect to σ. Each $p_i^{\alpha_i}$ is both q-monic and q-shift homogeneous. Note that there are many different ways to express $p_i^{\alpha_i}$ in (1), because

$$p_i^{\alpha_i} = \left(p_i^{\sigma^{\ell}}\right)^{\sigma^{-\ell}\alpha_i}$$

for all $\ell \in \mathbb{Z}$. Nonetheless, the q-monic and q-shift homogeneous components $p_i^{\alpha_i}$'s are uniquely determined by f, since $C[x]$ is a unique factorization domain. So we call (1) the *q-shift-homogeneous factorization* of f or *σ-factorization* for short.

Let f be a nonzero rational function in $C(x)$, and p be a q-monic and irreducible polynomial of positive degree. Then there exists a unique element $\alpha \in \mathbb{Z}[\sigma, \sigma^{-1}]$ such that f/p^{α} has no factor σ-equivalent to p. We call α the *σ-exponent* of p in f. In addition, the multiplicity of x in f is also called the σ-exponent of x in f.

Note that a rational function K is σ-reduced if and only if, for every nonconstant, q-monic and irreducible polynomial p, the nonzero coefficients of the σ-exponent of p in K have the same sign. The next proposition describes a special property of σ-reduced rational functions and will be used to distinguish rational and irrational q-hypergeometric terms.

Proposition 1. *Let r be a σ-reduced rational function in $C(x)$. If $r = \sigma^k(f)/f$ for some $f \in C(x)$ and $k \in \mathbb{Z}$, then r is a power of q.*

Proof. The conclusion clearly holds if $k = 0$. Assume that k is nonzero and that the σ-factorization of f is given in (1). Suppose that $s > 0$. Then

$$r = q^{km} p_1^{\beta_1} \cdots p_s^{\beta_s},$$

where $m \in \mathbb{Z}$ and $\beta_i = \sigma^k \alpha_i - \alpha_i \neq 0$ for all i with $1 \leq i \leq s$. It follows that β_i must have both positive and negative coefficients. On the other hand, the coefficients of β_i are either all nonpositive or all nonnegative, because r is σ-reduced. This contradiction implies that $s = 0$, i.e., $r = q^{km}$. □

Corollary 2. *Let $T \in R$ be a q-hypergeometric term. Assume that K is a kernel of $\sigma(T)/T$. Then K is a power of q if and only if T is of the form cf for some $c \in C_R$ and $f \in C(x)$.*

Proof. Assume that $K = q^m$ for some integer m. Then $\sigma(T)/T = q^m \sigma(S)/S$, where S is the corresponding shell of $\sigma(T)/T$ with respect to K. It follows from the equality $q^m = \sigma(x^m)/x^m$ that $T/(x^m S)$ is a constant, say c, of the ring R. Thus $T = cx^m S$. Taking $f = x^m S$ yields the assertion. Conversely, assume that $T = cf$ with $c \in C_R$ and $f \in C(x)$. Then $\sigma(T)/T = \sigma(f)/f = K\sigma(S)/S$. Thus, $K = \sigma(r)/r$ with $r = f/S$, which belongs to $C(x)$. By Proposition 1, K is a power of q. □

Note that R can be chosen so that C_R coincides with the field C if C is further assumed to be algebraically closed. Indeed, with an algebraically closed field C, we are able to construct a Picard-Vessiot extension of $C(x)$ that having no new constants and containing all σ-hypergeometric terms that interest us (cf. [5, 10]).

4 Shell reduction

Let T be a q-hypergeometric term whose σ-quotient has a kernel K and the corresponding shell S. Then there exists another q-hypergeometric term H with σ-quotient K such that $T = SH$, which is called a *multiplicative decomposition* of T. We are going to reduce the shell S modulo $\operatorname{im}(\Delta_K)$ to a rational function r, which is minimal in some sense. The reduction leads to $T = \Delta(G) + rH$ for some q-hypergeometric term G. Some special properties of r and H will make it easy to decide the q-summability of T. We begin with a description on the properties that r should satisfy.

Definition 1. A nonzero and q-monic polynomial $f \in C[x]$ is called *q-shift-free* or *σ-free* for short if $\gcd(f, \sigma^i(f)) = 1$ for all nonzero integer i.

The reader may find a more general definition of q-shift free polynomials in [9].

Remark 3. For a nonzero polynomial with the σ-factorization given in (1), the polynomial is σ-free if and only if $m = 0$ and every α_i is a monomial in $\mathbb{N}[\sigma, \sigma^{-1}]$.

Definition 2. Let $K \in C(x)$ be a σ-reduced rational function whose numerator and denominator are u and v, respectively. A nonzero polynomial $f \in C[x]$ is said to be *strongly coprime with K* if $\gcd(\sigma^i(f), u) = \gcd(\sigma^{-i}(f), v) = 1$ for all $i \in \mathbb{N}$.

Remark 4. Let the σ-factorization of a nonzero polynomial f be given in (1). Assume that λ_i and μ_i are the σ-exponents of p_i in u and v, respectively, $i = 1, \ldots, s$. Then f is strongly coprime with K if and only if

$$\operatorname{tdeg}(\alpha_i) > \operatorname{hdeg}(\lambda_i) \quad \text{and} \quad \operatorname{hdeg}(\alpha_i) < \operatorname{tdeg}(\mu_i) \quad \text{for all } i \text{ with } 1 \leq i \leq s.$$

The next lemma is used to verify the minimality of our additive decomposition in the sequel.

Lemma 3. *Let $K \in C(x)$ be a σ-reduced rational function with numerator u and denominator v. Let $g \in C(x)$ with denominator d, which is σ-free and strongly coprime with K. If there exist two rational functions $\tilde{g}$ and r such that*

$$v(g - \tilde{g}) - (u\sigma(r) - vr) \in C[x, x^{-1}], \tag{2}$$

then the degree of d is no more than that of the denominator of $\tilde{g}$.

Proof. Let $\tilde{d}$ be the denominator of $\tilde{g}$. There is nothing to show if $d \in C$. Now assume that $d \notin C$ and consider a nontrivial irreducible factor $p \in C[x]$ of d with multiplicity k. Since d is strongly coprime with K, it is coprime with v. Since d is σ-free, it suffices to prove that $\sigma^\ell(p)^k \mid \tilde{d}$ for some $\ell \in \mathbb{Z}$. To this end, we let e be the denominator of r. Suppose that p^k does not divide $\tilde{d}$, otherwise we have done. Then it follows from (2) that either $p^k \mid e$ or $p^k \mid \sigma(e)$.

If $p^k \mid e$, then there is an integer $\ell \geq 1$ such that $\sigma^{\ell-1}(p)^k \mid e$ but $\sigma^\ell(p)^k \nmid e$. Moreover, $\sigma^\ell(p)^k \mid \sigma(e)$. On the other hand, $\sigma^\ell(p) \nmid d$ because d is σ-free; and $\sigma^\ell(p) \nmid u$ because d is strongly coprime with K. Thus, (2) implies that $\sigma^\ell(p)^k \mid \tilde{d}$.

If $p^k \mid \sigma(e)$, then there is an integer $\ell \leq -1$ such that $\sigma^\ell(p)^k \mid e$ but $\sigma^{\ell-1}(p)^k \nmid e$, i.e., $\sigma^\ell(p)^k \nmid \sigma(e)$. Observe that $\sigma^\ell(p) \nmid d$ because d is σ-free, and $\sigma^\ell(p) \nmid v$ because d is strongly coprime with K. Thus, $\sigma^\ell(p)^k \mid \tilde{d}$ by (2). $\square$

Now, we describe how to perform shell reduction "locally".

Lemma 4. *Let $K \in C(x)$ be a σ-reduced rational function with numerator u and denominator v. Let $f \in C(x)$ be a nonzero rational function with denominator p^α, where $p \in C[x]$ is nonconstant, q-monic and irreducible and $\alpha \in \mathbb{N}[\sigma, \sigma^{-1}]$. Assume that λ and μ are the σ-exponents of p in u and v, respectively. Then we have the following two assertions.*

(i) If $\mu = 0$, then, for every integer ℓ with $\ell \geq \operatorname{hdeg}(\alpha)$, there exist k in $\mathbb{N}$ and a, b in $C[x]$ such that

$$f \equiv \frac{a}{p^{k\sigma^\ell}} + \frac{b}{v} \mod \operatorname{im}(\Delta_K). \tag{3}$$

(ii) If $\lambda = 0$, then, for every integer ℓ with $\ell \leq \operatorname{tdeg}(\alpha)$, there exist k in $\mathbb{N}$ and a, b in $C[x]$ such that (3) *also holds.*

Proof. If $\alpha = 0$, then $f \in C[x]$. So we just need to set $k = 0$, $a = f$ and $b = 0$, and assume that α is nonzero in the rest of the proof.

(i) Assume that $\alpha = \sum_{i=m}^{n} k_i \sigma^i$, where $m \leq n$, $k_i \in \mathbb{N}$ and $k_m k_n \neq 0$. Since p is q-monic and irreducible, the polynomials $p^{\sigma^m}, p^{\sigma^{m+1}}, \ldots, p^{\sigma^n}$ are pairwise coprime. Then we have a partial fraction decomposition $f = \sum_{i=m}^{n} f_i$, where f_i is either zero or has the denominator $p^{k_i \sigma^i}$ for all i with $m \leq i \leq n$.

Assume that f_i is nonzero. By the forward congruence in Corollary 1, for every integer ℓ with $\ell \geq n$, there exists $g_i \in C[x]$ such that

$$f_i \equiv \frac{g_i}{p^{k_i \sigma^\ell} \prod_{j=0}^{\ell-i-1} \sigma^j(v)} \mod \operatorname{im}(\Delta_K).$$

It follows from $\mu = 0$ that p^{σ^ℓ} is coprime with any q-shifts of v. Then there exist two polynomials $a_i, \tilde{a}_i$ in $C[x]$ such that

$$f_i \equiv \frac{a_i}{p^{k_i \sigma^\ell}} + \frac{\tilde{a}_i}{\prod_{j=0}^{\ell-i-1} \sigma^j(v)} \mod \operatorname{im}(\Delta_K). \tag{4}$$

Applying the first congruence in Lemma 2 to the second summand in the right-hand side of the above congruence, we find $b_i \in C[x]$ such that

$$f_i \equiv \frac{a_i}{p^{k_i \sigma^\ell}} + \frac{b_i}{v} \mod \operatorname{im}(\Delta_K).$$

Summing up all these congruences yields

$$f \equiv \frac{a}{p^{k\sigma^\ell}} + \frac{b}{v} \mod \operatorname{im}(\Delta_K),$$

where $a, b \in C[x]$ and $k \in \mathbb{N}$ with $k \leq \max(k_m, k_{m+1}, \ldots, k_n)$.

(ii) The congruence (3) can be proved by a similar argument, in which we use the backward congruence in Corollary 1 and the second congruence in Lemma 2. Moreover, $\lambda = 0$ implies that p^{σ^ℓ} is coprime with any q-shifts of u. Therefore, a partial fraction decomposition similar to (4) holds, in which $\sigma^j(v)$ is replaced with $\sigma^{-j}(u)$ and j ranges from 1 to $i - \ell$. $\square$

The above lemma leads to a key step for the shell reduction.

Corollary 3. *Let K, f, p and α be the same as those in Lemma 4. Then there exist two polynomials $a, b \in C[x]$ and a monomial $\beta \in \mathbb{N}[\sigma, \sigma^{-1}]$ such that*

$$f \equiv \frac{a}{p^\beta} + \frac{b}{v} \mod \operatorname{im}(\Delta_K). \tag{5}$$

Moreover, p^β is both σ-free and strongly coprime with K.

Proof. Let λ and μ be the σ-exponents of p in u and v, respectively. Then either λ or μ is zero since K is σ-reduced.

First, assume that p^α is strongly coprime with K. Set $\ell = \operatorname{hdeg}(\alpha)$ when $\mu = 0$ or $\ell = \operatorname{tdeg}(\alpha)$ when $\lambda = 0$. By Lemma 4, the congruence (5) holds in which $\beta = k\sigma^\ell$ is a monomial. Hence, p^β is σ-free and strongly coprime with K.

Second, assume that p^α is not strongly coprime with K. Then either $\operatorname{tdeg}(\alpha)$ is no greater than $\operatorname{hdeg}(\lambda)$ or $\operatorname{hdeg}(\alpha)$ is no smaller than $\operatorname{tdeg}(\mu)$.

If $\operatorname{tdeg}(\alpha) \le \operatorname{hdeg}(\lambda)$. then neither α nor λ equals zero. Thus, $\mu = 0$ because K is σ-reduced. Set $\ell = \max(\operatorname{hdeg}(\alpha), \operatorname{hdeg}(\lambda)+1)$. By Lemma 4 (i), the congruence (5) holds, in which β is a monomial. Consequently, p^β is σ-free. Moreover, it is strongly coprime with K, as $\operatorname{tdeg}(\beta) > \operatorname{hdeg}(\lambda)$ and $\operatorname{hdeg}(\beta) < \operatorname{tdeg}(\mu) = +\infty$.

If $\operatorname{hdeg}(\alpha) \ge \operatorname{tdeg}(\mu)$, then neither α nor μ is zero. So $\lambda = 0$. The congruence (5) holds by Lemma 4 (ii), in which ℓ is set to be $\min(\operatorname{tdeg}(\alpha), \operatorname{tdeg}(\mu)-1)$. $\square$

The main result of this section is given below.

Theorem 1. *Let $K \in C(x)$ be a σ-reduced rational function whose numerator and denominator are u and v, respectively. For every rational function $f \in C(x)$, there exists a proper rational function $g \in C(x)$ and a Laurent polynomial $h \in C[x, x^{-1}]$ such that*

$$f \equiv g + \frac{h}{v} \mod \operatorname{im}(\Delta_K) \tag{6}$$

with the property that the denominator of g is σ-free and strongly coprime with K. Moreover, the denominator of g is of minimal degree in the sense that if there exists another pair $(\tilde{g}, \tilde{h})$ with $\tilde{g} \in C(x)$ and $\tilde{h} \in C[x, x^{-1}]$ such that

$$f \equiv \tilde{g} + \frac{\tilde{h}}{v} \mod \operatorname{im}(\Delta_K) \tag{7}$$

then the degree of the denominator of g is no greater than that of $\tilde{g}$. In particular, $g = 0$ if $f \in \operatorname{im}(\Delta_K)$.

Proof. Let $cx^m \prod_{i=1}^s p_i^{\alpha_i}$ be the σ-factorization of the denominator of f, as described in (1). Then a partial fraction decomposition of f is

$$f = a + \sum_{i=1}^{s} f_i,$$

where a is a Laurent polynomial and f_i is proper with denominator $p_i^{\alpha_i}$ for $i = 1$, $\ldots$, s. By Corollary 3, we have, for all i with $1 \le i \le s$,

$$f_i \equiv \frac{a_i}{p_i^{\beta_i}} + \frac{b_i}{v} \mod \operatorname{im}(\Delta_K),$$

where $a_i, b_i \in C[x]$ and $p_i^{\beta_i}$ is σ-free and strongly coprime with K. Then (6) holds with $g = \sum_{i=1}^{s} a_i/p_i^{\beta_i}$ and $h = va + \sum_{i=1}^{s} b_i$. Note that the irreducible polynomials $p_1, \ldots, p_s$ are q-monic and mutually inequivalent with respect to σ. Thus, the denominator of g is σ-free. It is clearly strongly coprime with K. Moreover, g is proper since the forward and backward congruences do not change the degrees.

It remains to verify that the degree of the denominator of d is minimal. Assume that there exist $\tilde{g} \in C(x)$ and $\tilde{h} \in C[x, x^{-1}]$ such that (7) holds. By (6) and (7), there exists a rational function $r \in C(x)$ such that

$$g + \frac{h}{v} = \frac{u}{v}\sigma(r) - r + \tilde{g} + \frac{\tilde{h}}{v}.$$

Clearing the denominators in this equality, we see that $\deg(d)$ is no greater than the degree of the denominator of $\tilde{g}$ by Lemma 3.

Assume that $f \in \operatorname{im}(\Delta_K)$. Then $f \equiv 0 \mod \operatorname{im}(\Delta_K)$. Taking $\tilde{g} = \tilde{h} = 0$ in (7) implies that $g \in C[x]$ by the minimality of $\deg(d)$. Since g is proper, it is zero. □

Remark 5. On the way to compute g and h in (6), we can obtain another rational function r such that

$$f = \Delta_K(r) + g + \frac{h}{v},$$

because all the reductions are based on the forward and backward congruences, which can be easily transformed into equalities, as described in Remark 1.

Let us translate Theorem 1 into the q-hypergeometric setting. This leads to a q-analogue of Proposition 3.3 in [6].

Corollary 4. *Let T be a q-hypergeometric term whose σ-quotient has a kernel K with the denominator v. Then we have the following assertions.*

(i) There exist two rational functions $r, g \in C(x)$, a Laurent polynomial $h \in C[x, x^{-1}]$, and a q-hypergeometric term H with $\sigma(H)/H = K$ such that

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right)H.$$

Moreover, g is proper, and its denominator is σ-free, strongly coprime with K.

(ii) If T is q-summable, then $g = 0$.

Proof. (i) Let S be the shell of $\sigma(T)/T$ corresponding to K. By Theorem 1,

$$S \equiv g + \frac{h}{v} \mod \operatorname{im}(\Delta_K), \tag{8}$$

where g is a proper rational function whose denominator is σ-free and strongly coprime with K, and h belongs to $C[x, x^{-1}]$. Consequently, there exists $r \in C(x)$ such that

$$S = K\sigma(r) - r + g + \frac{h}{v}.$$

Set $H = T/S$. Then $\sigma(H)/H = K$. It follows that

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right) H.$$

(ii) Assume now that T is q-summable, that is, SH is q-summable, which is equivalent to the fact $S \in \operatorname{im}(\Delta_K)$. Therefore, $g = 0$ by Theorem 1 and (8). □

The shell reduction for q-hypergeometric terms renders us an additive decomposition for q-rational functions.

Corollary 5. *For $T \in C(x)$, there exist $f, g \in C(x)$ and $c \in C$ such that*

$$T = \Delta(f) + g + c, \tag{9}$$

where g is a proper rational function with σ-free denominator d. Moreover, if there exist $\tilde{f}, \tilde{g}$ in $C(x)$ and $\tilde{c}$ in C such that

$$T = \Delta(\tilde{f}) + \tilde{g} + \tilde{c}, \tag{10}$$

then $\deg(d)$ *is no greater than the degree of the denominator of $\tilde{g}$. In particular, T is q-summable if and only if $g = c = 0$.*

Proof. Let K be a kernel of $\sigma(T)/T$. By Corollary 2, $K = q^m$ for some $m \in \mathbb{Z}$. So we may take 1 as the denominator of K. By Corollary 4 (i), there exist a rational function $r \in C(x)$, a proper rational function $s \in C(x)$ with σ-free denominator d, a Laurent polynomial $t \in C[x, x^{-1}]$, and a q-hypergeometric term H with σ-quotient q^m such that $T = \Delta(rH) + (s + t)H$. Thus, H belongs to $C(x)$, and, consequently, is equal to $c' x^m$ for some $c' \in C$. It follows that

$$T = \Delta(c' r x^m) + c' s x^m + c' t x^m,$$

Moreover, we can split $c' s x^m$ into the sum of a Laurent polynomial and a proper rational function g whose denominator is equal to d. So $T - \Delta(c' r x^m) - g \in C[x, x^{-1}]$, which, together with the fact

$$c_i x^i = \Delta\left(\frac{c_i}{q^i - 1} x^i\right) \quad \text{for all } i \in \mathbb{Z} \text{ with } i \neq 0 \text{ and } c_i \in C,$$

implies that (9) holds.

It follows from (9) and (10) that

$$g - \tilde{g} - \left(\sigma(f - \tilde{f}) - (f - \tilde{f})\right) \in C[x, x^{-1}].$$

Setting $u = 1$ and $v = 1$ in Lemma 3, we see that $\deg(d)$ is no greater than the degree of the denominator of $\tilde{g}$.

If both g and c in (9) are equal to zero, then T is clearly q-summable. Conversely, assume that T is q-summable. Then one can choose both $\tilde{g}$ and $\tilde{c}$ to be zero in (10).

It follows from the minimality of $\deg(d)$ that $g = 0$. Consequently, c is q-summable, and, thus, $c = 0$. $\square$

Corollary 5 is derived from the shell reduction. It may also be obtained by translating the results in [1] directly into the q-case.

At last, we turn the proof of Theorem 1 into an algorithm, named after *ShellReduction*. To this end, we need to assume that one can factor univariate polynomials over C in the rest of this paper. For example, C is an algebraic number field over $\mathbb{Q}$ or the field of rational functions in several variables other than x over $\mathbb{Q}$.

ShellReduction. Given a σ-reduced rational function $K \in C(x)$ whose numerator and denominator are u and v, respectively, and a nonzero rational function $f \in C(x)$, compute two rational functions $r, g \in C(x)$ and a Laurent polynomial $h \in C[x, x^{-1}]$ such that

$$f = \Delta_K(r) + g + \frac{h}{v},$$

and g is proper whose denominator is σ-free and strongly coprime with K.

1. Compute the σ-factorization $cx^m p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ of the denominator of f, where $c \in C \setminus \{0\}$, $m \in \mathbb{N}$, $p_1, \ldots, p_s$ are q-monic and irreducible in $C[x] \setminus C$, inequivalent to each other with respect to σ, and $\alpha_1, \ldots, \alpha_s$ belong to $\mathbb{N}[\sigma, \sigma^{-1}] \setminus \{0\}$.
2. Compute the partial fraction decomposition of f to get
$$f = a + \sum_{i=1}^{s} f_i$$
where $a \in C[x, x^{-1}]$ and f_i is proper with denominator $p_i^{\alpha_i}$ for $i = 1, \ldots, s$.
3. For i from 1 to s do the following. Apply Corollary 3 to f_i and find a rational function r_i in $C(x)$, a_i, b_i in $C[x]$ and a monomial β_i in $\mathbb{N}[\sigma, \sigma^{-1}]$ such that
$$f_i = \Delta_K(r_i) + \frac{a_i}{p_i^{\beta_i}} + \frac{b_i}{v}.$$
4. Set
$$r := \sum_{i=1}^{s} r_i; \quad g := \sum_{i=1}^{s} \frac{a_i}{p^{\beta_i}}; \quad h := va + \sum_{i=1}^{s} b_i$$
and return.

Example 1. Let $(q;q)_n := \prod_{i=1}^{n}(1 - q^i)$ be a q-Pochhammer symbol and

$$T(n) = \frac{(q;q)_n}{1 + q^n},$$

which is a q-hypergeometric term with $\sigma(T(n)) = T(n+1)$ and $q^n = x$. Then the σ-quotient of T has a kernel $K = -qx + 1$ and the corresponding shell $S = 1/(x+1)$. Shell reduction yields

$$S = \Delta_K(0) + \frac{1}{x+1} + \frac{0}{v}$$

where $v = 1$ and the second summand is nonzero. By Corollary 4 (ii), T is non-summable.

Example 2. Let $T = -q^{n+1}(q;q)_n$. Then a kernel K of the σ-quotient of T is equal to $-q^2x+q$ and the corresponding shell S is equal to 1. According to the shell reduction algorithm, $S = \Delta_K(0) + 0 + 1/v$, where $v = 1$. But T is q-summable as $T = \Delta\left((q;q)_n\right)$.

The above example illustrates that the shell reduction cannot decide q-summability completely. One way to proceed is to find a Laurent polynomial solution of an auxiliary first-order linear q-recurrence equation, as in the usual shift case [2, 3]. We show how this can be avoided in the next section.

5 Reduction for Laurent polynomials

Corollary 4 (ii) provides us with a necessary condition on the summability of q-hypergeometric terms. To obtain a necessary and sufficient condition, we confine the numerator h in (6) into a finite-dimensional linear subspace over C. This idea was first presented in [4], and has been extended in various ways [6, 7, 8].

To deal with Laurent polynomials whose tail and head degrees are arbitrary, we shall first reduce negative powers and then positive ones. To guarantee the termination of our reduction, we introduce the notion of *reduction index*, abbreviated as rind. For a Laurent polynomial $f \in C[x,x^{-1}]$,

$$\operatorname{rind}(f) := \begin{cases} \operatorname{tdeg}(f) & \text{if } \operatorname{tdeg}(f) < 0 \\ \operatorname{hdeg}(f) & \text{if } \operatorname{tdeg}(f) \geq 0 \end{cases}$$

Note that $\operatorname{rind}(0)$ is equal to $-\infty$, and that nonzero Laurent polynomials with distinct reduction indices are linearly independent over C.

Lemma 5. *Let $K \in C(x)$ be a σ-reduced rational function with numerator u and denominator v. Define*

$$\begin{aligned} \phi_K : C[x,x^{-1}] &\longrightarrow C[x,x^{-1}] \\ f \quad &\mapsto \; u\sigma(f) - vf. \end{aligned}$$

Then we have the following assertions.

(i) The C-linear map ϕ_K is injective if K is not a power of q.
(ii) Define

$$\operatorname{im}(\phi_K)^{\top} = span_C\left\{x^d \mid d \neq rind(p) \; for\ all \; p \in \operatorname{im}(\phi_K)\right\}.$$

Then $C[x,x^{-1}] = \mathrm{im}\,(\phi_K) \oplus \mathrm{im}(\phi_K)^\top$.

Proof. (i) Assume that K is not a power of q. If $\phi_K(f) = 0$ for some $f \in C[x,x^{-1}]$, then either $f = 0$ or $v/u = \sigma(f)/f$. The latter implies that K is a power of q by Proposition 1, which is impossible. So $f = 0$, that is, ϕ_K is injective.

(ii) By the definition of $\mathrm{im}(\phi_K)^\top$, we have $\mathrm{im}(\phi_K) \cap \mathrm{im}(\phi_K)^\top = \{0\}$ and there is a Laurent polynomial

$$f_m \in \mathrm{im}(\phi_K) \cup \mathrm{im}(\phi_K)^\top$$

such that $\mathrm{rind}(f_m) = m$ for every integer $m \in \mathbb{Z}$. Set $B = \{f_m \mid m \in \mathbb{Z}\}$, which consists of linearly independent Laurent polynomials. It suffices to show that B is a C-basis of $C[x,x^{-1}]$. Let g be a nonzero Laurent polynomial whose reduction index equals r.

Case 1. Assume that $r \geq 0$. Then g is a C-linear combination of $f_0, f_1, \ldots, f_r$.

Case 2. Assume that $r < 0$. Then there is a C-linear combination h of f_r, f_{r+1}, $\ldots$, f_{-1} such that $g - h$ is of nonnegative tail degree. It follows from case 1 that $g - h$ belongs to the span of B over C, and so does g.

Hence, B is a C-basis of $C[x,x^{-1}]$. □

The map ϕ_K defined in the above lemma is called the *reduction map for Laurent polynomials with respect to* K or *the LP-reduction map* for short when K is clear from the context, and $\mathrm{im}(\phi_K)^\top$ is called the *standard complement of* $\mathrm{im}(\phi_K)$. The LP-reduction map is equal to the restriction of $v\Delta_K$ on $C[x,x^{-1}]$, where v is the denominator of K.

The importance of standard complements is described in the next lemma.

Lemma 6. *Let K be a σ-reduced rational function with denominator v, and ϕ_K be the LP-reduction map. If $g \in \mathrm{im}(\phi_K)^\top$ and $g/v \in \mathrm{im}(\Delta_K)$, then g is equal to zero.*

Proof. Assume that $g \in \mathrm{im}(\phi_K)^\top$ and $g/v \in \mathrm{im}(\Delta_K)$. It follows from $g/v \in \mathrm{im}(\Delta_K)$ that there exists $f \in C(x)$ such that

$$u\sigma(f) - vf = g, \tag{11}$$

where u is the numerator of K. Suppose that f is not a Laurent polynomial. Then its denominator d has a nonconstant, irreducible and q-monic factor p. Let α be the σ-exponent of p in d with tail and head degrees k and ℓ, respectively. Then $\sigma^k(p)$ is not a divisor of $\sigma(d)$. It follows from (11) that $\sigma^k(p)$ divides v. Similarly, $\sigma^{\ell+1}(p)$ divides u, as it is a divisor of the denominator of $\sigma(f)$ but not a divisor of d. We have reached a contradiction with the assumption that K is σ-reduced. Thus, f is a Laurent polynomial. Hence, $g \in \mathrm{im}(\phi_K) \cap \mathrm{im}(\phi_K)^\top$, which implies that $g = 0$. □

Next, we develop an algorithm for projecting a Laurent polynomial onto the image of an LP-reduction map and its standard complement, respectively.

Let $K \in C(x)$ be σ-reduced but not a power of q. By Lemma 5 (i), $\mathrm{im}\,(\phi_K)$ has a C-basis $\left\{\phi_K(x^k) \mid k \in \mathbb{Z}\right\}$. From this basis, we can compute another C-basis whose elements have distinct reduction indices, which will be referred to as an *echelon basis*. With such a basis, we can project a Laurent polynomial by linear elimination. To this end, we let $K = u/v$ with $u, v \in C[x]$ and $\gcd(u,v) = 1$. Set

$$u = \sum_{i=0}^{d} u_i x^i \quad \text{and} \quad v = \sum_{i=0}^{d} v_i x^i,$$

where the u_i and v_i belong to C for all i with $0 \leq i \leq d$ and $d = \max(\deg(u), \deg(v))$. So at least one of u_d and v_d is nonzero. Moreover, either u_0 or v_0 is nonzero because $\gcd(u,v) = 1$. For all $k \in \mathbb{Z}$,

$$\phi_K\left(x^k\right) = \left(u_0 q^k - v_0\right) x^k + \left(u_1 q^k - v_1\right) x^{k+1} + \cdots + \left(u_d q^k - v_d\right) x^{k+d}. \tag{12}$$

We make a case distinction to construct respective echelon bases of $\operatorname{im}(\phi_K)$ and $\operatorname{im}(\phi_K)^\top$. In what follows, $\mathbb{Z}^-$ stands for the set of all negative integers.

Case 1. Assume that, for all $k \in \mathbb{Z}^-$, $u_0 q^k - v_0$ is nonzero. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to k for all $k \in \mathbb{Z}^-$ by (12). To compute the reduction index of $\phi_K\left(x^k\right)$ for $k \in \mathbb{N}$, we need to consider two subcases.

Case 1.1. Assume further that $u_d q^k - v_d \neq 0$ for all $k \in \mathbb{N}$. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to $d+k$ for all $k \in \mathbb{N}$ by (12). So the images of distinct powers of x under ϕ_K have distinct reduction indices, and thus form an echelon basis of $\operatorname{im}(\phi_K)$. It follows that $\operatorname{im}(\phi_K)^\top$ has a basis $\left\{1, x, \ldots, x^{d-1}\right\}$.

Case 1.2. Assume that $u_d q^\ell - v_d = 0$ for some $\ell \in \mathbb{N}$. The integer ℓ is unique, because q is not a root of unity. Similar to case 1.1, the reduction index of $\phi_K\left(x^k\right)$ is equal to $d+k$ for all $k \in \mathbb{N}$ with $k \neq \ell$. However, the reduction index of $\phi_K\left(x^\ell\right)$ is less than $d+\ell$. Eliminating $x^{d+\ell-1}$, $x^{d+\ell-2}$, $\ldots$, x^d from $\phi_K\left(x^\ell\right)$ successively by $\phi_K\left(x^{\ell-1}\right)$, $\phi_K\left(x^{\ell-2}\right)$, $\ldots$, $\phi_K\left(x^0\right)$, we find $c_{\ell-1}, c_{\ell-2}, \ldots, c_0 \in C$ and $r \in C[x]$ with $\deg(r) < d$ such that

$$\phi_K\left(x^\ell\right) = \sum_{i=0}^{\ell-1} c_i \phi_K(x^i) + r.$$

Note that r is a nonzero polynomial in $C[x]$, because $\phi_K\left(x^0\right), \ldots, \phi_K\left(x^{\ell-1}\right), \phi_K\left(x^\ell\right)$ are linearly independent over C. Thus, an echelon basis of $\operatorname{im}(\phi_K)$ is

$$\{r\} \cup \left\{\phi_K(x^k) \mid k \in \mathbb{Z} \text{ with } k \neq \ell\right\}.$$

Consequently, $\operatorname{im}(\phi_K)^\top$ has a C-basis

$$\left\{1, x, \ldots, x^{\deg(r)-1}, x^{\deg(r)+1}, \ldots, x^{d-1}, x^{d+\ell}\right\}.$$

Case 2. There exists a negative integer ℓ such that $u_0 q^\ell - v_0 = 0$. Then the integer ℓ is unique. By (12), the reduction index of $\phi_K\left(x^k\right)$ is equal to k for all $k \in \mathbb{Z}^-$ with $k \neq \ell$, while the reduction index of $\phi_K\left(x^\ell\right)$ is greater than ℓ and less than d. Eliminating $x^{\ell+1}, x^{\ell+2}, \ldots, x^{-1}$ from $\phi_K\left(x^\ell\right)$ successively, we find $c_{\ell+1}, c_{\ell+2}, \ldots, c_{-1} \in C$ and a nonzero polynomial $r_\ell \in C[x]$ with $\deg(r_\ell) < d$ such that

$$\phi_K\left(x^\ell\right) = \sum_{i=\ell+1}^{-1} c_i \phi_K\left(x^i\right) + r_\ell. \tag{13}$$

We also need to consider two subcases.

Case 2.1. Assume that $u_d q^k - v_d \neq 0$ for all $k \in \mathbb{N}$. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to $d+k$ for all $k \in \mathbb{N}$. So $\operatorname{im}(\phi_K)$ has an echelon basis

$$\left\{\phi_K\left(x^k\right) \mid k \in \mathbb{Z}^-, k \neq \ell\right\} \cup \{r_\ell\} \cup \left\{\phi_K\left(x^k\right) \mid k \in \mathbb{N}\right\}.$$

Consequently, $\operatorname{im}(\phi_K)^\top$ has a C-basis

$$\left\{x^\ell, 1, x, \ldots, x^{\deg(r_\ell)-1}, x^{\deg(r_\ell)+1}, \ldots, x^{d-1}\right\}.$$

Case 2.2. Assume that $u_d q^m - v_d = 0$ for some $m \in \mathbb{N}$. The integer m is again unique. By (12), the reduction index of $\phi_K\left(x^k\right)$ is $d+k$ for all $k \in \mathbb{N}$ with $k \neq m$, and the reduction index of $\phi_K\left(x^m\right)$ is a nonnegative integer less than $d+m$. So there exist $c_0, c_1, \ldots, c_{m-1} \in C$ and a nonzero polynomial $r_m \in C[x]$ with degree less than d such that

$$\phi_K\left(x^m\right) = \sum_{j=0}^{m-1} c_j \phi_K(x^j) + r_m. \tag{14}$$

Moreover, r_ℓ and r_m are linearly independent over C, for otherwise, the images

$$\phi_K\left(x^\ell\right), \phi_K\left(x^{\ell+1}\right), \ldots, \phi_K\left(x^{-1}\right), \phi_K(x^0), \phi_K(x), \ldots, \phi_K\left(x^{m-1}\right)$$

would be linearly dependent, a contradiction to Lemma 5 (i). Set $p_\ell = r_\ell$ and

$$p_m = \begin{cases} r_m & \text{if } \deg(r_\ell) \neq \deg(r_m) \\[2ex] \operatorname{lc}(r_m) r_\ell - \operatorname{lc}(r_\ell) r_m & \text{otherwise.} \end{cases}$$

Then p_ℓ, p_m and r_ℓ, r_m span the same linear subspace over C, but the degrees of p_ℓ and p_m are distinct elements in $\{0, 1, \ldots, d-1\}$. It follows that $\operatorname{im}(\phi_K)$ has an echelon basis

$$\left\{\phi_K\left(x^k\right) \mid k \in \mathbb{Z}^-, k \neq \ell\right\} \cup \{p_\ell, p_m\} \cup \left\{\phi_K\left(x^k\right) \mid k \in \mathbb{N}, k \neq m\right\}$$

and that $\operatorname{im}(\phi_K)^\top$ has a C-basis

$$\left\{x^\ell, 1, x, \ldots, x^{d-1}, x^{d+m}\right\} \setminus \left\{x^{\deg(p_\ell)}, x^{\deg(p_m)}\right\}.$$

The above detailed case distinction leads to two interesting consequences. The first one tells us that all elements in a standard complement are “sparse” Laurent polynomials, as their numbers of terms are bounded.

Proposition 2. *Let $K = u/v$, where $u, v \in C[x]$ and $\gcd(u, v) = 1$. Assume that K is σ-reduced and not a power of q. Then the standard complement of the LP-reduction map is of dimension* $\max(\deg(u), \deg(v))$.

Proof. It is immediate from the last conclusions in cases 1.1, 1.2, 2.1 and 2.2. □

Example 3. Let $u = x^3 + q^{11}$ and $v = q^{20}x^3 + 1$. Then $K = u/v$ is σ-reduced. Note that $u_0 q^{-11} - v_0 = 0$, and $\mathrm{lc}(u)q^{20} - \mathrm{lc}(v) = 0$, where u_0 and v_0 are the coefficients of x^0 in u and v, respectively. It follows from (13) and (14) that r_{-11} and r_{20} are polynomials of degrees 1 and 2, respectively. As $\mathrm{rind}(r_{-11}) \neq \mathrm{rind}(r_{20})$, we set $p_{-11} = r_{-11}$ and $p_{20} = r_{20}$. Thus, $\mathrm{im}(\phi_K)^\top = \mathrm{span}_C\{x^{-11}, 1, x^{23}\}$ by the last conclusion made in case 2.2. Consequently, every element of $\mathrm{im}(\phi_K)^\top$ has at most three terms.

Second, the case distinction enables us to project a Laurent polynomial onto $\mathrm{im}(\phi_K)$ and $\mathrm{im}(\phi_K)^\top$, respectively.

LPReduction. Given a Laurent polynomial $h \in C[x, x^{-1}]$, compute $a \in C[x, x^{-1}]$ and $b \in \mathrm{im}(\phi_K)^\top$ such that $h = \phi_K(a) + b$.

1. If $h = 0$, then set $a = 0$ and $b = 0$; return.
2. Find the subset $\{f_1, \ldots, f_s\} \subseteq C[x, x^{-1}]$ consisting of the preimages of all polynomials in an echelon basis of $\mathrm{im}(\phi_K)$ whose reduction indices are no more than $\mathrm{hdeg}(h)$ and no less than $\mathrm{tdeg}(h)$.
3. Order the echelon basis such that
$$\mathrm{rind}(\phi_K(f_1)) < \cdots < \mathrm{rind}(\phi_K(f_t)) < \mathrm{rind}(\phi_K(f_{t+1})) < \ldots < \mathrm{rind}(\phi_K(f_s)),$$
with $\mathrm{rind}(\phi_K(f_t)) < 0$ and $\mathrm{rind}(\phi_K(f_{t+1})) \geq 0$.
4. For $i = 1, 2, \ldots, t$, perform linear elimination to find $c_i \in C$ such that
$$g := h - \sum_{i=1}^{t} c_i \phi_K(f_i) \in C[x] + \mathrm{im}(\phi_K)^\top.$$
5. For $i = s, s-1, \ldots, t+1$, perform linear elimination to find $c_i \in C$ such that
$$b := g - \sum_{i=t+1}^{s} c_i \phi_K(f_i) \in \mathrm{im}(\phi_K)^\top.$$
6. Set $a := \sum_{i=1}^{s} c_i f_i$ and return a, b.

The truncated echelon basis in step 2 can be easily constructed according to the above case distinction. In step 4, we eliminate all negative power of x in h except those appearing in $\mathrm{im}(\phi_K)^\top$. In step 5, we eliminate all positive powers of x in g except those appearing in $\mathrm{im}(\phi_K)^\top$. Then the resulting Laurent polynomial b is the projection of h on $\mathrm{im}(\phi_K)^\top$. Tracing back the two elimination processes, we obtain the preimage of the projection of h on $\mathrm{im}(\phi_K)$.

In summary, we have the following additive decomposition for irrational q-hypergeometric terms.

Theorem 2. *Let T be an irrational q-hypergeometric term whose σ-quotient has a kernel K. Let u and v be the numerator and denominator of K, respectively, and ϕ_K be the LP-reduction map. Then the following four assertions hold.*

(i) There is an algorithm to compute a q-hypergeometric term H, two rational functions $f, g \in C(x)$ and a Laurent polynomial $p \in \operatorname{im}(\phi_K)^\top$ such that

$$T = \Delta(fH) + \left(g + \frac{p}{v}\right)H, \tag{15}$$

where the σ-quotient of H is equal to K, g is proper, and its denominator is σ-free and strongly coprime with K.

(ii) p has at most $\max(\deg(u), \deg(v))$ many nonzero terms.

(iii) If there exist $\tilde{f}, \tilde{g} \in C(x)$ and $\tilde{p} \in C[x, x^{-1}]$ such that

$$T = \Delta(\tilde{f}H) + \left(\tilde{g} + \frac{\tilde{p}}{v}\right)H, \tag{16}$$

then the degree of the denominator of g is no greater than that of $\tilde{g}$.

(iv) T is q-summable if and only if both g and p are equal to zero.

Proof. (i) Let S be the shell of $\sigma(T)/T$ with respect to K and $H = T/S$. Applying the shell reduction algorithm to S, we obtain $r, g \in C(x)$ and $h \in C[x, x^{-1}]$ such that

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right)H, \tag{17}$$

where g is a proper rational function whose denominator is σ-free and strongly coprime with K.

The LP-reduction algorithm computes two Laurent polynomials a and p such that $h = \phi_K(a) + p$ and $p \in \operatorname{im}(\phi_K)^\top$. Hence, $h = u\sigma(a) - va + p$. It follows that

$$\frac{h}{v} = K\sigma(a) - a + \frac{p}{v} = \Delta_K(a) + \frac{p}{v},$$

which, together with (17), implies that (15) holds (setting $f = r + a$).

(ii) It is immediate from Proposition 2.

(iii) Assume that both (15) and (16) hold. Then

$$S = \Delta_K(f) + g + \frac{p}{v} = \Delta_K(\tilde{f}) + \tilde{g} + \frac{\tilde{p}}{v}.$$

It follows from Theorem 1 that the degree of the denominator of g is no greater than that of $\tilde{g}$.

(iv) If both g and p are equal to zero, then T is clearly q-summable. Conversely, assume that T is q-summable. By (17) and Theorem 1, $g = 0$. Hence, $(p/v)H$ is also q-summable. In other words, $p/v \in \operatorname{im}(\Delta_K)$. By Lemma 6, $p = 0$. $\square$

We now present an algorithm to decompose a q-hypergeometric term into a q-summable term and a non-summable one, which determines q-summability without solving any auxiliary q-recurrence equation explicitly. The algorithm, named q-MAP, is a q-analogue of the modified Abramov-Petkovšek algorithm.

q-MAP. Given a q-hypergeometric term T, compute two q-hypergeometric terms T_1 and T_2 such that $T = \Delta(T_1) + T_2$ with the property that T_2 is minimal in the sense of Theorem 2 (iii) and T is q-summable if and only if T_2 is zero.

1. Compute a kernel K and the corresponding shell S of $\sigma(T)/T$. Set v to be the denominator of K. Set $H = T/S$.
2. Apply **ShellReduction** to S to find $f, g \in C(x)$ and $h \in C[x, x^{-1}]$ such that
$$T = \Delta(fH) + \left(g + \frac{h}{v}\right) H.$$
3. If $K = q^m$ for some integer m, then compute $a \in C[x, x^{-1}]$ and $c \in C$ such that
$$hx^m = \Delta(a) + c,$$
according to the proof of Corollary 5 (In this case, $v = 1$ and $H = x^m$). Set
$$T_1 := fx^m + a \quad \text{and} \quad T_2 := gx^m + c;$$
and return.
4. If $K \neq q^m$ for any integer m, then apply **LPReduction** to h and find $a \in C[x, x^{-1}]$ and $b \in \mathrm{im}(\phi_K)^\top$ such that
$$h = \phi_K(a) + b,$$
where ϕ_K is the LP-reduction map. Set
$$T_1 := (f + a)H \quad \text{and} \quad T_2 := \left(g + \frac{b}{v}\right) H;$$
and return.

Example 4. Consider the same term in Example 2. By shell reduction,
$$S = \Delta_K(0) + 0 + \frac{1}{v},$$
where $v = 1$. Then apply the LP reduction on the numerator 1 to get
$$S = \Delta_K(-(qx)^{-1}) + 0 + 0,$$
which implies that T is q-summable and $T = \Delta((q;q)_n)$.

6 Experimental results

We have implemented our q-analogue of the modified Abramov-Petkovšek reduction in the computer algebra system MAPLE 18, and compared with two analogues of Gosper's algorithm in [12] and [14], respectively.

The first analogue, named q-Gosper's algorithm [1], has three steps:

1. Compute a q-Gosper form (a,b,c) of the σ-quotient of the input term.
2. Estimate degree bounds for a Laurent polynomial solution of a q-analogue of Gosper's equation in the form

$$a\sigma(z) - \sigma^{-1}(b)z = c. \tag{18}$$

3. Compute the Laurent polynomial solution by solving a linear system over C.

It takes little time to compute q-Gosper forms and estimate the head and tail degree bounds of Laurent polynomial solutions of (18). So most of the time is spent on solving a linear system over C.

The other q-analogue is named after *qTelescope* by the authors of [14]. It uses greatest factorial factorization to compute three polynomials P,Q,R such that P is coprime with both Q and R, and Q is coprime with every positive q-shift of R, and then computes a polynomial solution of a variant of (18) in the form

$$Q\sigma(Z) - RZ = P \tag{19}$$

by solving a linear system over C. Our implementation of the qTelescope algorithm is based on the description given in [14].

The test suite was generated by

$$T := \frac{a}{p_1\ \sigma^{\ell_1}(p_1)\ p_2\ \sigma^{\ell_2}(p_2)} \prod_{k=1}^{n-1} \frac{u_1 u_2}{v_1 v_2}, \tag{20}$$

where

(i) $a, p_1, p_2 \in \mathbb{Q}(q)[q^n]$ are random with $\deg(a) = 30$, $\deg(p_1) = \deg(p_2) = d$, where q is transcendental over $\mathbb{Q}$, and n is an integral variable;
(ii) $u_1, u_2, v_1, v_2 \in \mathbb{Q}(q)[q^n]$ are random whose degrees are all equal to 1;
(iii) $\ell_1, \ell_2 \in \mathbb{N}$.

In all the examples given above, the q-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is equal to $\max(\ell_1,\ell_2)$. All timings are measured in seconds on an OS X computer with 16GB 1600MHz DDR3 and 2.5 GHz Intel Core i7 processors.

Table 1 contains the timings of the q-Gosper's algorithm (q-Gosper), qTelescope algorithm (qTelescope) and the q-analogue of the modified Abramov-Petkovšek reduction (q-MAP) for input T given in (20) with different choices of ℓ_1 and ℓ_2. In general, randomly-generated terms in the form (20) are non-summable. In this

[1] We thank Dr. Haitao Jin for sending us his maple scripts on q-Gosper's algorithm

case, both q-Gosper's algorithm and qTelescope algorithm return a message "non-summable"; while the q-MAP algorithm not only determines the non-summability, but also presents an additive decomposition. The experimental results illustrate that the q-MAP algorithm outperforms the two q-analogues of Gosper's algorithm when the degree d is equal to five and the q-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is greater than one.

Algorithm \ (ℓ_1,ℓ_2)	(2,2)	(2,3)	(3,4)	(4,4)	(4,5)	(5,5)	(5,10)	(10,10)
q-Gosper	18.615	45.718	99.296	125.136	271.063	328.437	3332.533	7707.963
qTelescope	23.413	46.952	120.891	120.495	173.247	405.994	2541.752	7574.879
q-MAP	3.355	4.626	8.182	10.181	12.829	15.611	47.104	90.532

Table 1 Non-summable case: $d = 5$ and (ℓ_1,ℓ_2) varies

Table 2 contains the timings of the three algorithms for input T given in (20), in which the q-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is equal to one. One can show that the estimation on the degree bound in the second step of q-Gosper's algorithm has already implies that (18) has no Laurent polynomial solution when $d \geq 15$. Thus, q-Gosper's algorithm determines the non-summability of T instantly, and so does the algorithm qTelescope.

Algorithm \ d	1	5	10	15	20	25	30
q-Gosper	2.3	2.1	0.6	0.03	0.04	0.04	0.04
qTelescope	3.1	3.8	1.7	0.5	0.8	1.3	1.8
q-MAP	3.9	1.6	5.5	32.3	150.4	517.1	1523.1

Table 2 Non-summable case: $(\ell_1,\ell_2) = (1,1)$ and d varies

Table 3 contains the timings of the three algorithms for input $\Delta(T)$, where T is the same as in Table 1. So all the input terms are q-summable. Both q-Gosper and qTelescoper are either faster than or comparable with the q-MAP reduction when ℓ_1 and ℓ_2 take small values. In this case, the q-dispersion of the denominator of the input rational function in the shell reduction is less than or equal to 10. When the q-dispersion is more than 10, the q-modified Abramov-Petkovšek reduction outperforms both of the q-analogues.

When the degree bound estimates are loose in q-Gosper's algorithm, the q-modified Abramov-Petkovšek reduction is markedly superior to q-Gosper's algorithm, as illustrated in the next example.

Example 5. Let

$$f := \frac{(x^3+q^{11})(q^5x^5+q^3x^3+2)(x-1)(q^{10}x-1)(x-2)(q^{20}x-2)}{(q^{20}x^3+1)(x^5+x^3+2)(qx-1)(q^{11}x-1)(qx-2)(q^{21}x-2)}$$

Algorithm \ (ℓ_1,ℓ_2)	(2,2)	(5,5)	(5, 10)	(10, 10)	(10, 20)	(20, 20)	(20, 30)	(30, 30)
q-Gosper	0.992	3.288	8.637	13.481	76.261	104.880	383.131	479.860
qTelescope	4.719	13.464	29.991	42.078	147.516	256.348	923.929	1823.568
q-MAP	3.288	11.546	18.939	22.203	40.440	44.703	90.023	88.876

Table 3 Summable case: $d = 5$ and (ℓ_1,ℓ_2) varies

be the σ-quotient of some q-hypergeometric term T.

Applying q-Gosper's algorithm to the term $\Delta(T)$, we compute a q-Gosper form (a,b,c) of the σ-quotient of $\Delta(T)$, where

$$a = (x^3+q^{11})(x-1)(x-2) \quad \text{and} \quad b = (q^{23}x^3+1)(q^{12}x-1)(q^{22}x-2).$$

As the ratio of the leading coefficients, as well as that of the tailing coefficients, is a power of q, the estimates on both head and tail degrees are not sharp. Indeed, the bounds on head and tail degrees estimated in q-Gosper's algorithm are 52 and -11, respectively. But a Laurent polynomial solution of the q-Gosper equation is of head degree 33 and tail degree 0. It takes about 35 seconds to find the indefinite sum of $\Delta(T)$.

Similarly, 63 is the degree bound for a polynomial solution of (19) in the algorithm qTelescope, and a polynomial solution of (19) is equal to $x^{11}p$, where p is a polynomial of degree 33. It takes about 9 seconds to find the indefinite sum of $\Delta(T)$.

On the other hand, q-MAP takes less than 0.3 second to find the indefinite sum, although the case 2.2 happens in the LP-reduction.

Acknowledgements. We thank the anonymous referee for helpful comments and valuable references.

References

1. Sergei A. Abramov. The rational component of the solution of a first order linear recurrence relation with a rational right hand side. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 15(4):1035–1039, 1090, 1975.
2. Sergei A. Abramov and Marko Petkovšek. Minimal decomposition of indefinite hypergeometric sums. In *ISSAC 2001—Proceedings of the 26th International Symposium on Symbolic and Algebraic Computation*, pages 7–14 (electronic). ACM, New York, 2001.
3. Sergei A. Abramov and Marko Petkovšek. Rational normal forms and minimal decompositions of hypergeometric terms. *J. Symbolic Comput.*, 33(5):521–543, 2002. Computer algebra (London, ON, 2001).
4. Alin Bostan, Shaoshi Chen, Frédéric Chyzak, Ziming Li, and Guoce Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*, pages 77–84. ACM, New York, 2013.

5. Manuel Bronstein, Ziming Li, and Min Wu. Picard-Vessiot extensions for linear functional systems. In *ISSAC 2005—Proceedings of the 30th International Symposium on Symbolic and Algebraic Computation*, pages 68–75. ACM, New York, 2005.
6. Shaoshi Chen, Hui Huang, Manuel Kauers, and Ziming Li. A modified Abramov-Petkovšek reduction and creative telescoping for hypergeometric terms. In *ISSAC 2015—Proceedings of the 40th International Symposium on Symbolic and Algebraic Computation*, pages 117–124. ACM, New York, 2015.
7. Shaoshi Chen, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for algebraic functions. In *ISSAC 2016—Proceedings of the 41st International Symposium on Symbolic and Algebraic Computation*, page to appear. ACM, New York, 2016.
8. Shaoshi Chen, Mark van Hoeij, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for Fuchsian D-finite functions. *J. Symbolic Comput.* to appear.
9. William Y. C. Chen, Qing-Hu Hou, and Yan-Ping Mu. Applicability of the q-analogue of Zeilberger's algorithm. *J. Symbolic Comput.*, 39(2):155–170, 2005.
10. Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.
11. Michael Karr. Summation in finite terms. *J. Assoc. Comput. Mach.*, 28(2):305–350, 1981.
12. Tom H. Koornwinder. On Zeilberger's algorithm and its q-analogue. *J. Comput. Appl. Math.*, 48(1):91–111, 1993.
13. Peter Paule. Greatest factorial factorization and symbolic summation. *J. Symbolic Comput.*, 20(3):235–268, 1995.
14. Peter Paule and Axel Riese. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping. In *Special functions, q-series and related topics (Toronto, ON, 1995)*, volume 14 of *Fields Inst. Commun.*, pages 179–210. Amer. Math. Soc., Providence, RI, 1997.
15. Carsten Schneider. Simplifying Sums in $\Pi\Sigma^*$-Extensions. J. Algebra Appl. 6(3), pp. 415-441. 2007.