Complete Reduction for Derivatives in a Transcendental Liouvillian Extension*

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Abstract

Transcendental Liouvillian extensions are differential fields, in which one can model polylogarithmic, hyperexponential, and trigonometric functions, logarithmic integrals, and their (nested) rational expressions. For such an extension, we construct, over the subfield of constants, a complement of the subspace of derivatives, and develop an algorithm that decomposes any element of the field into the sum of a derivative and a component lying in the complement. Consequently, an element is a derivative if and only if its complementary component vanishes. Moreover, the algorithm enables us to determine elementary integrability over the extension by computing parametric logarithmic parts, and leads to a reduction-based approach to constructing telescopers for elements in the extension, provided that an a priori order bound is given.

Keywords: Additive decomposition, Complete reduction, Creative telescoping, Elementary integration, In-field integration, Liouvillian extension, Risch equation, Symbolic integration

1 Introduction

Symbolic integration as a classical topic of computer algebra is to express indefinite integrals in closed forms. Historical developments and fundamental results on this topic are surveyed in [33].

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From a computational perspective, the most systematic advances are due to Risch [34, 35] and numerous researchers who have refined, enhanced, extended, and implemented Risch's algorithm [36, 18, 38, 19, 8, 9, 10, 32].

For an elementary function, Risch's algorithm determines whether it has an elementary integral, and computes such an integral if there exists one.

Example 1.1. *Let* $f(x) = \dfrac{x \log(x)^3 + 1}{x \log(x)}$ *and* $g(x) = \dfrac{x \log(x)^3 + 1}{(x+3)\log(x)}$. *By Risch's algorithm,*

$$\int f \, dx = x \log(x)^2 - 2x \log(x) + 2x + \log(\log(x)) \quad \text{and} \quad \int g \, dx = \int \frac{x \log(x)^3 + 1}{(x+3)\log(x)} \, dx.$$

The algorithm yields an elementary integral of f, and reveals that g has no elementary integral.

The most basic routine in Risch's algorithm is the Hermite-Ostrogradsky reduction for rational functions [28, 31]. We refer to [10, Chapter 2] and [25, Chapter 11] for its modern presentations.

In the rest of this section, let C be a field of characteristic zero.

Example 1.2. *For every element $f \in C(x)$, the Hermite-Ostrogradsky reduction computes two elements $g, r \in C(x)$ such that $f = g' + r$, where $'$ stands for the usual derivation d/dx, and r is an x-proper fraction with a squarefree denominator. Moreover, r is unique, and $f \in C(x)'$ if and only if $r = 0$, where $C(x)'$ stands for the C-subspace consisting of derivatives in $C(x)$.*

Set $S_x := \{s \in C(x) \mid s \text{ is } x\text{-proper and has a squarefree denominator}\}$. Then S_x is a C-subspace and $C(x) = C(x)' \oplus S_x$ by the Hermite-Ostrogradsky reduction. The projection from $C(x)$ to S_x with respect to the direct sum is a complete reduction for $C(x)'$ in the sense of [29, Definition 5.67], which is recalled below.

Definition 1.3. *Let V be a C-linear space and U be a subspace. A linear operator $\Phi : V \to V$ is called a* complete reduction *for U if $v - \Phi(v) \in U$ for all $v \in V$ and* $\ker(\Phi) = U$.

A straightforward linear algebra argument reveals that a C-linear operator Φ on V is a complete reduction for U if and only if $V = U \oplus \operatorname{im}(\Phi)$ and Φ is the projection from V onto $\operatorname{im}(\Phi)$ with respect to this direct sum. In particular, all complete reductions are idempotent.

In the present paper, the subspace U in the above definition is typically the image of some linear operator on V. We therefore adopt the following definition for convenience.

Definition 1.4. *Let V be a C-linear space and $\mathcal{L}$ be a linear operator on V. Another linear operator Φ on V is called a* complete reduction for the pair $(V, \mathcal{L})$ *if Φ is a complete reduction for* $\operatorname{im}(\mathcal{L})$. *In other words,* $V = \operatorname{im}(\mathcal{L}) \oplus \operatorname{im}(\Phi)$ *and Φ is the projection from V onto* $\operatorname{im}(\Phi)$ *with respect to this direct sum.*

The next definition describes how a complete reduction decomposes an element additively.

Definition 1.5. *Let Φ be a complete reduction for the pair $(V, \mathcal{L})$ and let $u, v \in V$ be such that $v = \mathcal{L}(u) + \Phi(v)$. We call $\Phi(v)$ the* remainder of v with respect to Φ, *and $(u, \Phi(v))$ a* reduction pair of v with respect to Φ *(or an* R-pair with respect to Φ *for brevity).*

Remark 1.6. *Given a linear operator $\mathcal{L}$ on a linear space V, we say that* there is an algorithm for constructing a complete reduction for $(V, \mathcal{L})$ *if we can fix a complement W of* $\operatorname{im}(\mathcal{L})$ *in V and develop an algorithm that, for every $v \in V$, computes $u \in V$ and $w \in W$ such that $v = \mathcal{L}(u) + w$. Such an algorithm induces a complete reduction for $(V, \mathcal{L})$ by mapping v to w.*

The Hermite-Ostrogradsky reduction in Example 1.2 not only induces a complete reduction for $(C(x), ')$, but also yields an algorithm for computing the corresponding R-pairs. Based on this reduction, an algorithm is developed to construct telescopers for bivariate rational functions (see [3]). The algorithm separates the computation of telescopers from the computation of certificates. This is desirable in the typical situation where we are only interested in telescopers. Such an advantage has motivated a rapid development of complete reductions in various settings.

Example 1.7. *Let h be a nonzero rational function in $C(x)$, and V be the space spanned by the hyperexponential function $\exp(\int h\, dx)$ over $C(x)$. The Hermite reduction for hyperexponential functions developed in [4] gives a complete reduction for $(V, d/dx)$.*

Recall that a *derivation* $' : R \to R$ on a commutative ring R is an additive map satisfying the usual Leibniz's rule: $(ab)' = a'b + ab'$ for all $a, b \in R$. The pair $(R, ')$ is called a *differential ring* and a *differential field* if R is a field. The set $\{c \in R \mid c' = 0\}$ forms a subring whose elements are called *constants*. This subring of constants becomes a subfield if R itself is a field.

Let $(R, ')$ be a differential ring whose subring C of constants is a field. Then R is a C-linear space. Complete reductions have been established for such differential rings with instances of R including the field of differential fractions [6], the field of algebraic functions [16], (sub)rings of D-finite functions [17, 39, 12], exponential towers [24] and primitive towers [21]. A complete reduction for the pair $(C(x), \mathcal{L})$ is developed in [5], where $\mathcal{L}$ denotes a linear differential operator. The reader is referred to [7, 15, 13] for complete reductions related to symbolic summation.

A complete reduction for $(R, ')$ decomposes $f \in R$ as $f = g' + r$ for some $g, r \in R$, where r is minimal with respect to a partial ordering defined in [21, Section 1]. In particular, f is a derivative in R if and only if $r = 0$. So a complete reduction for $(R, ')$ is an additive decomposition in the sense of [14, 22]. This offers an alternative, and potentially more informative, way to express integrals.

Example 1.8. *By viewing the functions f and g in Example 1.1 as elements of the differential field $\mathbb{Q}(x, \log(x))$ with derivation $' = d/dx$, the complete reduction in [21] yields*

$$f(x) = h(x)' + \frac{1}{x\log(x)} \quad \text{and} \quad g(x) = h(x)' + \frac{1}{(x+3)\log(x)} - \frac{3\log(x)^2}{x+3},$$

where $h(x) = x\log(x)^2 - 2x\log(x) + 2x$.

Note that $f(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder $1/(x\log(x))$, which has an elementary integral $\log(\log(x))$. By contrast, $g(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder, which is not elementarily integrable. The elementary integrability of such remainders in logarithmic extensions can be determined by computing parametric logarithmic parts (see [21, Theorem 5.3]). Unlike Risch's algorithm taken in Example 1.1, the complete reduction renders us an integrable part $h(x)$ of $g(x)$ in $\mathbb{Q}(x, \log(x))$ and a minimal non-integrable part with respect to the partial ordering defined in [21, Section 1].

Both decompositions can also be obtained from the algorithms in [14, 22].

We are interested in developing complete reductions on transcendental Liouvillian extensions. Such extensions are more general than transcendental elementary extensions (see [10, Definition 5.1.4]), and do not contain any complicated reduced elements (see [10, Definition 3.5.2]).

Definition 1.9. *A differential field $F_n = C(t_1, \ldots, t_n)$ with derivation $'$ is called a* transcendental Liouvillian extension *of C if*

(i) $t_1, \ldots, t_n$ are algebraically independent over C,

(ii) for every $i \in \{1, \ldots, n\}$, either $t_i' \in F_{i-1}$ or $t_i'/t_i \in F_{i-1}$, where $F_{i-1} = C(t_1, \ldots, t_{i-1})$,

(iii) C is the subfield of constants in F_n.

We call t_i a primitive *(resp.* hyperexponential*)* generator *if $t_i' \in F_{i-1}$ (resp. $t_i'/t_i \in F_{i-1}$). Moreover, F_n is called a* primitive *(resp.* hyperexponential*)* tower *if all of the t_i's are primitive (resp. hyperexponential).*

Due to the presence of hyperexponential generators, a more general inductive framework is indispensable (see Example 3.1). A similar phenomenon also occurs in the inductive proof of the main theorem in [34]. In fact, the theorem has two assertions. The first is of interest in elementary integration, and the second one serves to facilitate the proof. Within the second assertion, the (parametric) Risch differential equation

$$y' + hy = \sum_{i=1}^{m} c_i g_i \tag{1}$$

appears in the literature for the first time. We write h here instead of the symbol f used in [34], because f often stands for an arbitrary element of a differential field in the sequel. In (1), h and the g_i's belong to a differential field F, and the c_i's are constants to be determined.

To develop complete reductions for symbolic integration, we define Risch operators using the homogeneous part of (1).

Definition 1.10. *Let $(F, ')$ be a differential field and $h \in F$. The map*

$$\begin{array}{rccc} \mathcal{R}_h : & F & \to & F \\ & y & \mapsto & y' + hy \end{array}$$

is called the Risch operator associated to *h.*

Risch operators are linear over the subfield of constants in F. In particular, $\mathcal{R}_0$ is the derivation operator on F.

Assume that there is a complete reduction Φ_h for $(F, \mathcal{R}_h)$, where h is given on the left-hand side of (1). Then each of the g_i's on the right-hand side of (1) has an R-pair $(p_i, \Phi_h(g_i))$ with respect to Φ_h. Therefore, (1) can be rewritten as $\mathcal{R}_h(y) = \sum_{i=1}^{m} c_i \left(\mathcal{R}_h(p_i) + \Phi_h(g_i)\right)$, which is equivalent to

$$\mathcal{R}_h\left(y - \sum_{i=1}^{m} c_i p_i\right) = \sum_{i=1}^{m} c_i \Phi_h(g_i).$$

Since $\operatorname{im}(\mathcal{R}_h) \cap \operatorname{im}(\Phi_h) = \{0\}$, there exists an element $y \in F$ such that (1) holds for some constants $c_1, \ldots, c_m$ if and only if $\sum_{i=1}^{m} c_i \Phi_h(g_i) = 0$ and $\mathcal{R}_h\left(y - \sum_{i=1}^{m} c_i p_i\right) = 0$. The first constraint leads to a linear system over the subfield of constants in F, and the second amounts to determining the kernel of $\mathcal{R}_h$. In other words, the complete reduction finds all solutions of (1) in F by solving a linear algebraic system and $y' + hy = 0$ in F.

Let F_n be a transcendental Liouvillian extension of C, and h be any element of F_n. The main result of this paper is Theorem 7.1, in which an algorithm is presented for constructing a complete reduction for $(F_n, \mathcal{R}_h)$ for every $h \in F_n$. In doing so, we generalize the complete reduction in [21] published in the Proceedings of ISSAC 2025 from primitive towers to general transcendental Liouvillian extensions. The generalization also includes the complete reduction in Example 1.7 and that on exponential towers in [24] as special cases. In particular, setting $h := 0$ yields a complete reduction Φ_0 for derivatives in F_n.

The new complete reduction enables us to straightforwardly determine in-field integrability of functions that can be expressed as elements of a transcendental Liouvillian extension.

Example 1.11. *Let*

$$f = \frac{x}{1+\exp(x)} \cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

It can be regarded as an element in the transcendental Liouvillian extension $\mathbb{C}(x,t,y)$ *of* $\mathbb{C}$*, where* $t = \exp(x)$ *and* $y = \exp\left(\frac{x}{1+\exp(x)}\right)$*. Indeed,* $f = \frac{xy}{1+t} \in \mathbb{C}(x,t,y)$*. We determine whether* f *has an integral in* $\mathbb{C}(x,t,y)$*.*

The complete reduction given by the algorithm outlined in Section 7 finds that

$$f = -\left(y + \frac{y}{t}\right)'$$

and thus

$$\int f\,dx = -\big(1+\exp(-x)\big)\cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

The AXIOM*-based computer algebra system* FRICAS 1.3.13 *returns the same integral (see [23]). But the* `int()` *command (with option* `method=_RETURNVERBOSE`*) in* MAPLE *2026 leaves the integral unevaluated, and so does the* `Integrate[]` *command in* MATHEMATICA *14.3.*

Computing this integral amounts to carefully handling a special case in Risch's algorithm, which corresponds to Proposition 6.12 (ii). Details are given in Example 7.4.

In [32, Chapters 3 and 4], Raab provides several building blocks that were either missing or incomplete within the description of Risch's algorithm in [10], and presents a method for computing elementary integrals over admissible differential fields subject to some mild restrictions. Such fields are more general than transcendental Liouvillian extensions. Raab's method needs all the ingredients from Risch's algorithm for the transcendental case. The complete reduction presented in this paper leads to an alternative algorithm for elementary integration over transcendental Liouvillian extensions. It is based on some special structure of remainders described in Proposition 8.8, and techniques in [32, Section 4.3.1] for dealing with the logarithmic derivative recognition problem in [10, Section 5.12] and parametric logarithmic derivative problem in [10, Section 7.3].

Besides elementary integration, our complete reduction may also be used for reduction-based creative telescoping in transcendental Liouvillian extensions, as illustrated below.

Example 1.12. *Let*

$$R(k) = \int_0^{\frac{\pi}{2}} \cos(2kx)\log(\sin(x))\;dx \quad \text{and} \quad I(k) = \int_0^{\frac{\pi}{2}} \sin(2kx)\log(\sin(x))\;dx.$$

By the complete reduction, we find that $(k+1)\mathcal{S}_k - k$ *is a minimal telescoper for both* $R(k)$ *and* $I(k)$*, where* $\mathcal{S}_k$ *stands for the shift operator* $k \mapsto k+1$*. Then the telescoper yields two linear recurrences:*

$$R(k+1) - \frac{k}{k+1}R(k) = 0 \quad \text{and} \quad I(k+1) - \frac{k}{k+1}I(k) = \frac{(-1)^k - 2k - 1}{4k(k+1)^2},$$

which help us express $R(k)$ *and* $I(k)$ *in closed forms. Details are given in Example 8.13.*

Preliminary experiments in MAPLE 2026 illustrated that our complete reduction outperformed the `int()` command in MAPLE 2026 for computing in-field integrals of derivatives of elements in certain transcendental elementary extensions. In particular, a significant speed-up was observed when integrands were derivatives of polynomials in the generators of such an extension (see Tables 5 and 7 in Section 9).

Let $F_n = C(t_1, \ldots, t_{n-1}, t_n)$ be a transcendental Liouvillian extension of C. Our construction of a complete reduction proceeds by induction on n: the base case $n = 0$ corresponds to the subfield of constants, and for the inductive step we set $F := C(t_1, \ldots, t_{n-1})$ and $t = t_n$. Then t is either primitive or hyperexponential over F.

Assume that there is an algorithm for constructing complete reductions for $(F, \mathcal{R}_\alpha)$ for all $\alpha \in F$. For an element $h \in F(t)$, the overall strategy for constructing a complete reduction for $(F(t), \mathcal{R}_h)$ is outlined in Figure 1, in which t-normalized elements, the companion operator $\mathcal{P}_h$, and echelon sequences are defined in Definitions 3.3, 3.5 and 2.7, respectively. The differential subring $F\langle t\rangle$ of $F(t)$ is defined in (2).

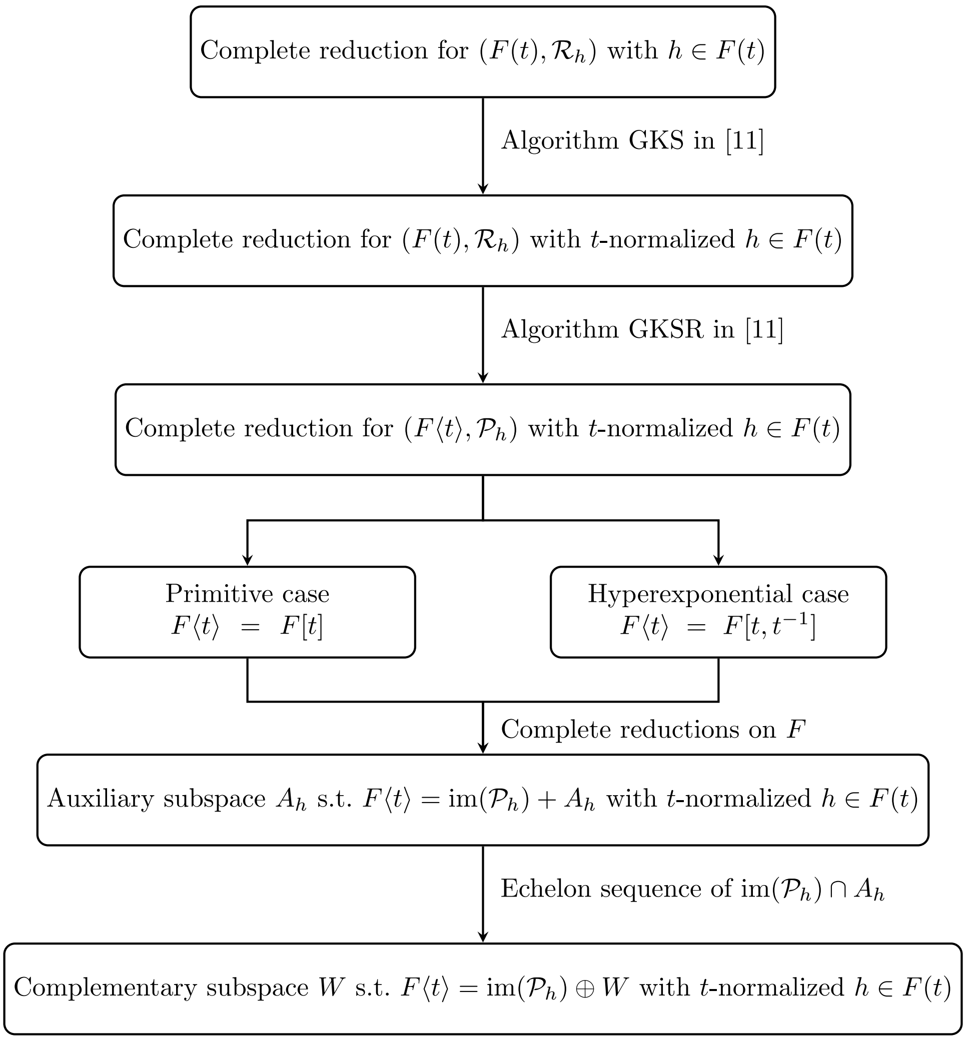

Figure 1: General outline of the complete reduction construction

We transform the problem of constructing a general complete reduction to that of constructing a complete reduction for $(F\langle t\rangle, \mathcal{P}_h)$ with h being t-normalized. The idea for such a transformation can be traced back to [19, Condition b) on page 905] and [10, Section 6.1], in which the problem of solving Risch differential equations in $F(t)$ is reduced to the corresponding problem in the differential subring $F\langle t\rangle$. Similar transformations are also given in [26, 11], which can be viewed, respectively, as a differential analogue and a generalization of the minimal decomposition for hypergeometric terms in [2, 1]. This step is based on Algorithms GKS and GKSR in [11]. Algorithm GKS decomposes an element of $F(t)$ as the sum of a t-normalized element and a logarithmic derivative, and Algorithm GKSR decomposes an element of $F(t)$ as the sum of an element of $\operatorname{im}(\mathcal{R}_h)$, an element of $F\langle t\rangle$ and a t-simple element (see Definition 2.1) under the assumption that $h \in F(t)$ is t-normalized.

Despite the substantial differences between primitive and hyperexponential generators, we develop a unified approach to constructing a complete reduction for $(F\langle t\rangle, \mathcal{P}_h)$. Our approach constructs an auxiliary subspace A_h such that $F\langle t\rangle = \operatorname{im}(\mathcal{P}_h) + A_h$, and computes an echelon sequence of $\operatorname{im}(\mathcal{P}_h) \cap A_h$. This sequence then enables us to find a complement of $\operatorname{im}(\mathcal{P}_h)$ in a dual manner. While related ideas have appeared in [4, Section 4.1], [24, Chapter 4] and [21, Section 3] in one way or another, a comprehensive generalization is needed in our setting, because $F(t)$ is more involved than the differential fields considered in those works. This step makes essential use of two algorithms in [32, Section 4.3.1]. One is for recognizing logarithmic derivatives in F, and the other is for solving the parametric logarithmic derivative problem in F.

The remainder of this paper is organized as follows. Section 2 introduces basic terminology from symbolic integration and provides a dual description of complementary subspaces. In Section 3, we recall the notion of t-normalized elements, define companion operators, and transform the problem of constructing complete reductions on a transcendental Liouvillian extension to that on a differential subring. Our induction hypothesis is given in Section 4. The most technical inductive steps are treated for primitive generators in Section 5, and for hyperexponential generators in Section 6. The induction is finalized in Section 7. Applications and computational benchmarks are presented in Sections 8 and 9, respectively. Section 10 contains concluding remarks. In the appendix, we describe algorithms derived from the constructive proofs given in Sections 5 and 6.

2 Preliminaries

This section consists of three parts. First, we introduce notation to be used in the sequel, along with fundamental terminology for symbolic integration. Next, we review several useful properties of residues relevant to elementary integration. Finally, we characterize certain complementary subspaces via kernels of linear functions. This characterization will enable us to construct complementary subspaces of infinite dimension in Sections 5 and 6.

2.1 Basic notation and terminology

Besides the usual notation in textbooks, we let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{N}^-$ be the sets of positive, nonnegative and negative integers, respectively. Set $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$, and $[n]_0 := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$. For an abelian group $(G, +, 0)$, we write $G^\times$ for $G \setminus \{0\}$. Let R be a commutative ring, $S \subset R$ and $x \in R$. The set $\{sx \mid s \in S\}$ is denoted by $S \cdot x$ or $x \cdot S$.

All fields appearing in the sequel are of characteristic zero. Let F be a field and $p \in F[t]^\times$. The degree and leading coefficient of p are denoted by $\deg_t(p)$ and $\operatorname{lc}_t(p)$, respectively. In addition, $\deg_t(0) := -\infty$ and $\operatorname{lc}_t(0) := 0$. We set $F[t]_{<d} := \{p \in F[t] \mid \deg_t(p) < d\}$ for all $d \in \mathbb{N}_0$.

The ring of Laurent polynomials in t over F is denoted by $F[t, t^{-1}]$. Let $q \in F[t, t^{-1}]$ be of the form $q_k t^k + q_{k-1} t^{k-1} + \cdots + q_{l+1} t^{l+1} + q_l t^l$, where $k, l \in \mathbb{Z}$, $k \geq l$, $q_k, q_{k-1}, \ldots, q_{l+1}, q_l \in F$, and $q_k q_l \neq 0$. The *head degree, head coefficient, tail degree and tail coefficient* of q are defined to be k, q_k, l and q_l, denoted by $\mathrm{hdeg}_t(q)$, $\mathrm{hc}_t(q)$, $\mathrm{tdeg}_t(q)$ and $\mathrm{tc}_t(q)$, respectively. In addition, $\mathrm{hdeg}_t(0) := -\infty$, $\mathrm{hc}_t(0) = 0$, $\mathrm{tdeg}_t(0) := \infty$, and $\mathrm{tc}_t(0) := 0$. Since

$$F[t, t^{-1}] = F[t] \oplus \left(t^{-1} \cdot F[t^{-1}]\right),$$

we set q^+ and q^- to be the respective projections of q to $F[t]$ and $t^{-1} \cdot F[t^{-1}]$. For $Q \subset F[t, t^{-1}]$ and $l \in \mathbb{Z}$, we denote by $Q_{>l}$ the set of elements in Q whose tail degrees are greater than l.

For an element f of $F(t)$, its numerator and denominator are denoted by $\mathrm{num}_t(f)$ and $\mathrm{den}_t(f)$, respectively. Moreover, $\mathrm{den}_t(f)$ is set to be monic. We say that f is *t-proper* if the degree of $\mathrm{num}_t(f)$ is less than that of $\mathrm{den}_t(f)$. For later convenience, the degree of f in t is defined to be the maximum of the degrees of its numerator and denominator, and is denoted by $\mathrm{Deg}_t(f)$. Note that $\deg_t(0) = -\infty$ but $\mathrm{Deg}_t(0) = 0$.

The subscript t will be omitted for brevity when the indeterminate t is clear from context.

Let $f \in F(t)^\times$, $a = \mathrm{num}(f)$ and $b = \mathrm{den}(f)$. For $p \in F[t]$ with $\deg(p) > 0$, the *order* of f at p, denoted by $\nu_p(f)$, is defined as follows: if p occurs in b with multiplicity $m > 0$ then $\nu_p(f) = -m$; otherwise, $\nu_p(f)$ is equal to the multiplicity of p in a. By convention, we set $\nu_p(0) = \infty$ for all $p \in F[t]$ with $\deg(p) > 0$. The *order of f at infinity*, denoted by $\nu_\infty(f)$, is defined as $\deg(b) - \deg(a)$.

Let $(F, {}')$ be a differential field. We denote $\{f' \mid f \in F\}$ by F', which is a linear space over the subfield of constants in F. An element of F is called a *logarithmic derivative* if it is equal to g'/g for some $g \in F^\times$. A differential field (E, δ) is called a *differential field extension* of $(F, {}')$ if F is a subfield of E and $\delta|_F = {}'$. The derivation δ will still be denoted by $'$ when there is no confusion.

Let $(E, {}')$ be a differential field extension of $(F, {}')$ and $u \in E$. Then u is said to be *primitive* (resp. *hyperexponential*) over F if $u' \in F$ (resp. $u \neq 0$ and $u'/u \in F$). An element u of E is said to be *logarithmic* (resp. *exponential*) over F if u' is a logarithmic derivative in F (resp. $u \neq 0$ and $u'/u \in F'$). If u is logarithmic (resp. exponential) over F, then it is primitive (resp. hyperexponential) over F. The converse is false. For example, the logarithmic integral $\mathrm{Li}(x)$, which equals $\int \frac{1}{\log x}$, is primitive but not logarithmic over $\mathbb{C}(x, \log(x))$, and $x \exp(x)$ is hyperexponential but not exponential over $\mathbb{C}(x)$.

An element t in a differential field extension of $(F, {}')$ is called a *monomial over F* if t is transcendental over F and $t' \in F[t]$. A monomial t over F is said to be *regular* if $F(t)$ and F have the same subfield of constants. By Definition 1.9, a transcendental Liouvillian extension $C(t_1, \ldots, t_n)$ is a differential field extension of C, in which t_i is a regular monomial over $C(t_1, \ldots, t_{i-1})$, and is either primitive or hyperexponential over the same field for all $i \in [n]$.

In the rest of this section, we let $(F, {}')$ be a differential field, C be its subfield of constants, and t be a monomial over F. Then $F[t]$ is a differential subring of $F(t)$. A nonzero polynomial $p \in F[t]$ is said to be *normal* (resp. *special*) if $\gcd(p, p') = 1$ (resp. $\gcd(p, p') = p$) (see [10, Definition 3.4.2]). The set of normal polynomials with positive degrees is denoted by N_t.

Let us recall the notion of t-simple elements in [11, Section 2.1].

Definition 2.1. *An element of $F(t)$ is t-simple if it is t-proper and has a normal denominator.*

All t-simple elements in $F(t)$ form an F-subspace, which is denoted by S_t. Let

$$F\langle t \rangle = \{f \in F(t) \mid \mathrm{den}(f) \text{ is special}\}. \tag{2}$$

Then $F\langle t \rangle$ is a differential subring (see [10, Corollary 4.4.1]). We remark that t-simple elements are not required to be t-proper in [10, Definition 3.5.2]. Our further requirement for t-properness results in $S_t \cap F\langle t \rangle = \{0\}$, which is necessary for the direct sum (6) in Section 3.

In what follows, we use lowercase Greek letters such as $\alpha, \beta, \gamma, \lambda, \mu, \sigma$ and τ to denote elements in F or those in the algebraic closure of F except coefficients given by polynomials in $F[t]$ and elements of C. On some occasions, lowercase Greek letters in Alegreya-LF style such as ξ, η and ω indicate elements of $F(t)$, which may possibly lie outside F.

Some basic facts about generators in a transcendental Liouvillian extension are collected in the following lemma for later references.

Lemma 2.2. *Let t be regular over F, $d \in \mathbb{Z}$, $\alpha \in F^{\times}$ and $p \in F[t]$.*

(i) If $d > 0$ and t is primitive over F, then $\left(\alpha t^d\right)' = \alpha' t^d + (\alpha d t')\, t^{d-1}$ whose degree is equal to d if α is not a constant, and $d-1$, otherwise.

(ii) If $d \neq 0$ and t is hyperexponential over F, then $\left(\alpha t^d\right)' = (\alpha' + \alpha d(t'/t))\, t^d$ whose head and tail degrees are both equal to d.

(iii) If t is primitive over F, then p is normal if and only if p is squarefree.

(iv) If t is hyperexponential over F, then p is normal if and only if p is squarefree and $t \nmid p$.

Proof. (i) Let m be the degree of $\left(\alpha t^d\right)'$. By Leibniz's rule, $\left(\alpha t^d\right)' = \alpha' t^d + (\alpha d t')\, t^{d-1}$. So $m = d$ if $\alpha' \neq 0$. Otherwise, $m = d - 1$ since t is both regular and primitive.

(ii) Similarly, Leibniz's rule implies that $\left(\alpha t^d\right)' = (\alpha' + \alpha d(t'/t))\, t^d$, which is nonzero by [10, Theorem 5.1.2]. Since $t'/t \in F$, both head and tail degrees of $\left(\alpha t^d\right)'$ are equal to d.

(iii) and (iv) hold by [10, Lemma 3.4.4, Theorems 5.1.1 and 5.1.2]. $\square$

We use sequences in algorithmic descriptions in order to avoid clumsy names of list operations. Let $L : l_1, \ldots, l_k$ be a sequence. Then l_i is denoted by $L[i]$ for all $i \in [k]$. The *length* of L is defined to be k and denoted by $\operatorname{len}(L)$. In the description of an algorithm, comments are placed between $(^* \cdots ^*)$. When outlining algorithms, we abbreviate "with respect to" and "such that" as "w.r.t." and "s.t.", respectively.

2.2 Residues and elementary integration

First, we recall the notion of residues given in [10, Definition 4.4.1].

Definition 2.3. *Let $p \in N_t$ and $V_p = \{f \in F(t) \mid \nu_p(f) \geq -1\}$. For an element $f \in V_p$, the* residue *of f at p, denoted by $\operatorname{residue}_p(f)$, is $\pi_p(fp/p')$, where π_p stands for the canonical homomorphism from the ring $\{q \in V_p \mid \nu_p(q) \geq 0\}$ to $F[t]/(p)$.*

Note that $\operatorname{residue}_p(f)$ is a congruence class. We identify $\operatorname{residue}_p(f)$ with the element of degree less than $\deg(p)$ in $\pi_p(fp/p')$, because we are mainly concerned with constant residues.

Remark 2.4. *The residue of $f \in S_t$ is well-defined at every element of N_t, as $\nu_p(f) \geq -1$ for all $p \in N_t$.*

Two well-known facts about residues are:

- all residues of an element in $F[t]$ are equal to zero;
- all residues of a logarithmic derivative in $F(t)$ are integers (see [10, Corollary 4.4.2 (iii)]).

The next two lemmas summarize some technical results scattered in [20], which will be useful for proving Theorem 8.7.

Lemma 2.5. *Let t be regular over F, $p \in F[t]^\times$ with $d = \deg(p)$ and $p_d = \operatorname{lc}(p)$.*

(i) If t is primitive over F, then $p'/p = s + p_d'/p_d$ for some $s \in S_t$.

(ii) If t is hyperexponential over F, then $p'/p = s + p_d'/p_d + dt'/t$ for some $s \in S_t$.

(iii) All residues of s appearing in (i) or (ii) are integers.

Proof. (i) Let $q = p_d^{-1}p$. Then q is monic. By [10, Exercise 3.1], we have $p'/p = p_d'/p_d + q'/q$. All irreducible factors of q are normal by Lemma 2.2 (iii). Applying Lemma 2.2 (i) and [10, Exercise 3.1] again gives $q'/q \in S_t$. The conclusion holds by setting $s := q'/q$.

(ii) There exists an integer $m \in \mathbb{N}_0$ and a monic polynomial $r \in F[t]$ with $t \nmid r$ such that $p = p_d t^m r$. All irreducible factors of r are normal by Lemma 2.2 (iv). The logarithmic derivative identity implies that

$$\frac{p'}{p} = \frac{p_d'}{p_d} + m\frac{t'}{t} + \frac{r'}{r}. \tag{3}$$

Set $s := r'/r - (d-m)t'/t$. Since $t'/t \in F$, it follows from [10, Exercise 3.1] and Lemma 2.2 (ii) that $s \in S_t$. Substituting $dt'/t + s$ for $mt'/t + r'/r$ in (3) yields that $p'/p = p_d'/p_d + dt'/t + s$.

(iii) The t-simple element s in (i) or (ii) is the difference of p'/p and an element in F. Since all residues of a logarithmic derivative are integers, so are the residues of s. □

Recall that a differential field extension $F(u_1, \ldots, u_m)$ is called an *elementary extension* of F if each u_j is either algebraic or logarithmic or exponential over F and the extension contains no new constants other than those in F. We say that an element of F has an *elementary integral* over F if it is a derivative in some elementary extension of F (see [10, Definition 5.1.4]).

Lemma 2.6. *Let t be regular over F and $f \in S_t^\times$. Assume that C is algebraically closed, and that t is either primitive or hyperexponential over F. If all residues of f belong to C, then f has an elementary integral over $F(t)$.*

Proof. Let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of $\operatorname{den}(f)$ in the algebraic closure $\overline{F}$ of F. Since all residues of f in $\overline{F}$ are constants, it follows from [20, Lemma 3.1 (i)] that

$$f = c_1\frac{(t-\alpha_1)'}{t-\alpha_1} + \cdots + c_k\frac{(t-\alpha_k)'}{t-\alpha_k} + p$$

for some $c_1, \ldots, c_k \in C$ and $p \in F[t]$. If t is primitive over F, then $p = 0$ by $f \in S_t$ and Lemma 2.2 (i). If t is hyperexponential over F, then $p = ct'/t$ for some $c \in C$ by $f \in S_t$ and Lemma 2.2 (ii). Thus, f has an elementary integral over $F(t)$. □

2.3 A dual presentation for complementary subspaces

Let V be a C-linear space. Every subspace of V can be written as the intersection of kernels of some linear functions on V. Such an intersection is understood as a dual presentation.

For a nonempty subset S of V, $\operatorname{span}_C(S)$ stands for the C-subspace spanned by S. The empty set spans the zero subspace $\{0\}$ by convention. Let Θ be a basis of V. For $\vartheta \in \Theta$, we denote by ϑ^* the linear function on V that maps ϑ to 1 and other elements of Θ to 0.

Definition 2.7. *Let V be a C-linear space, Θ be a basis of V, U be a subspace of V and $\mathbb{I} \subset \mathbb{N}_0$. A basis $\{u_i \mid i \in \mathbb{I}\}$ of U is called an* echelon basis of U with respect to Θ *if, for every $i \in \mathbb{I}$, there exists an element $\vartheta_i \in \Theta$ such that $u_i \notin \ker(\vartheta_i^*)$, and $u_j \in \ker(\vartheta_i^*)$ for all $j \in \mathbb{I}$ with $j < i$. In this case, we also call $\{u_i \mid i \in \mathbb{I}\}$ an* echelon basis of U with pivots $\{\vartheta_i \mid i \in \mathbb{I}\}$.

For example, an echelon basis $\{u_i \mid i \in \mathbb{N}_0\}$ of U with pivots $\{\vartheta_i \mid i \in \mathbb{N}_0\}$ is of the "upper triangular" form

$$\begin{aligned}
&\vdots \\
u_i &= c_i\vartheta_i + \sum_{\vartheta \notin \{\vartheta_i, \vartheta_{i+1}, \vartheta_{i+2}, \dots\}} c_\vartheta \vartheta \\
&\vdots \qquad \ddots \\
u_1 &= \qquad c_1\vartheta_1 + \sum_{\vartheta \notin \{\vartheta_1, \vartheta_2, \vartheta_3, \dots\}} c_\vartheta \vartheta \\
u_0 &= \qquad\qquad c_0\vartheta_0 + \sum_{\vartheta \notin \{\vartheta_0, \vartheta_1, \vartheta_2, \dots\}} c_\vartheta \vartheta,
\end{aligned}$$

where $i \in \mathbb{N}_0$, $c_i \in C^\times$, ϑ_i is the pivot of u_i, and $c_\vartheta \in C$, finitely many nonzero.

Lemma 2.8. *With the notation introduced in Definition 2.7, we further let $\{u_i \mid i \in \mathbb{I}\}$ be an echelon basis of U with pivots $\{\vartheta_i \mid i \in \mathbb{I}\}$. Then $V = U \oplus \left(\cap_{i\in\mathbb{I}} \ker(\vartheta_i^*)\right)$.*

Proof. Let $\widetilde{\Theta} = \Theta \setminus \{\vartheta_i \mid i \in \mathbb{I}\}$. Then $\cap_{i\in\mathbb{I}} \ker(\vartheta_i^*) = \operatorname{span}_C \widetilde{\Theta}$. The lemma follows from the observation that $\{u_i \mid i \in \mathbb{I}\} \cup \widetilde{\Theta}$ is a basis of V. □

When developing complete reductions, we face situations, in which a subspace U of V is the image of some linear operator, and it is difficult to directly construct its echelon basis. To construct a complement of U, we introduce an auxiliary subspace A such that $V = U + A$ and $U \cap A$ has an echelon basis. Then a complement of U can be obtained from the next lemma.

Lemma 2.9. *With the notation introduced in Definition 2.7, we further let A be a subspace of V such that $V = U + A$. If $U \cap A$ has an echelon basis with pivots $\{\vartheta_i \mid i \in \mathbb{I}\}$, then the intersection*

$$A \cap \Big(\bigcap_{i\in\mathbb{I}} \ker(\vartheta_i^*) \Big)$$

is a complement of U in V.

Proof. Let $W = \cap_{i\in\mathbb{I}} \ker(\vartheta_i^*)$. Then $V = U \cap A + W$ by Lemma 2.8. Since $U \cap A \subset A$, the modular law of subspaces gives $A = U \cap A + W \cap A$. Thus,

$$V = U + A = U + U \cap A + W \cap A = U + W \cap A.$$

The conclusion follows immediately from $U \cap A \cap W = \{0\}$. □

In this paper, reductions are typically carried out inside, or modulo, the image of a linear operator. This motivates us to define the notion of echelon sequences so as to find pre-images and images simultaneously during reduction.

Definition 2.10. *Let V be a C-linear space, $\mathcal{L}$ be a linear operator on V, Θ be a basis of V, and U be a subspace contained in $\operatorname{im}(\mathcal{L})$. Assume that $\{\mathcal{L}(v_i) \mid i \in \mathbb{I}\}$ is an echelon basis of U with pivots $\{\vartheta_i \mid i \in \mathbb{I}\}$, Then we call $\{(v_i, \mathcal{L}(v_i), \vartheta_i) \mid i \in \mathbb{I}\}$ an* echelon sequence of U with respect to $(V, \mathcal{L})$ and Θ.

Example 2.11. *Let $V = C[x]$ with a C-basis $\Theta = \{x^i \mid i \in \mathbb{N}_0\}$, and $\mathcal{L}$ be the linear operator that maps $p \in V$ to $(x^2+1)p$. Then $\operatorname{im}(\mathcal{L})$ has an echelon sequence $\left\{\left(x^i, x^{i+2} + x^i, x^{i+2}\right)\right\}_{i\in\mathbb{N}_0}$. By Lemma 2.8, a complement of $\operatorname{im}(\mathcal{L})$ in V is $\cap_{i\in\mathbb{N}_0} \ker(\vartheta_i^*)$ with $\vartheta_i = x^{i+2}$. This complement is equal to $C[x]_{<2}$.*

In the final part of this section, we make a preparation for computing R-pairs (see Definition 1.5). First, we recall the definition of effective bases in [21, Section 2.2].

Definition 2.12. *Let V be a C-linear space. A basis Θ of V is said to be* effective *if two algorithms are available: one takes every $v \in V^\times$ and computes an element $\vartheta \in \Theta$ such that $\vartheta^*(v) \neq 0$, and the other evaluates $\vartheta^*(v)$ for all $v \in V$ and $\vartheta \in \Theta$.*

The transcendental Liouvillian extension $C(t_1, \ldots, t_n)$ defined in Definition 1.9 admits an effective basis via successive irreducible partial fraction decompositions; details can be found in Algorithms 2.4, 2.6 and Remark 2.7 in [21, Section 2.2].

Lemma 2.13. *With the notation introduced in Definition 2.10, we further assume that Θ is effective. If $\{(v_i, \mathcal{L}(v_i), \vartheta_i) \mid i \in \mathbb{I}\}$ is an echelon sequence of U with respect to $(V, \mathcal{L})$ and Θ, then, for every $x \in V$, one can compute an element $v \in V$ and its image $\mathcal{L}(v) \in U$ (as a byproduct) such that $x - \mathcal{L}(v) \in \cap_{i \in \mathbb{I}} \ker(\vartheta_i^*)$.*

Proof. Since Θ is effective, one can express x as a linear combination of elements in Θ. Let Θ_x be the finite set of all pivots appearing in this linear combination. If $\Theta_x = \emptyset$, then letting $v = 0$ and thus $\mathcal{L}(v) = 0$ concludes the proof.

Assume that $\Theta_x \neq \emptyset$ and consider the maximal index j with $\vartheta_j \in \Theta_x$. Set $c_x := \vartheta_j^*(x)$ and $c_j := \vartheta_j^*(\mathcal{L}(v_j))$, which is nonzero. Define $y_j := c_x c_j^{-1} v_j$. Then $\mathcal{L}(y_j) = c_x c_j^{-1} \mathcal{L}(v_j) \in U$, and, by the maximality of j, we have $x - \mathcal{L}(y_j) \in \ker(\vartheta_i^*)$ for all $i \geq j$.

Now replace x by $x - \mathcal{L}(y_j)$ and repeat the argument. Then the pivot set of the new x does not contain any element ϑ_k with $k \geq j$. So after finitely many iterations we obtain a finite sum $v = \sum_l y_l$ and the corresponding $\mathcal{L}(v)$ with the required property. □

3 Normalization and companion operators

Throughout this section, we let $(F, ')$ be a differential field, C be the subfield of constants in F, and t be a monomial over F.

We initially sought to construct a complete reduction for $(F(t), ')$ under the assumption that there was a complete reduction for $(F, ')$. This assumption suffices when t is regular and primitive over F. However, it becomes insufficient when t is hyperexponential over F, as illustrated by the following example.

Example 3.1. *Let t be both regular and hyperexponential. Set $\alpha = t'/t$, which belongs to F. For an element $\beta \in F$, we determine whether $\beta t \in F(t)'$. By an order argument, $\beta t \in F(t)'$ if and only if $\beta t \in F[t, t^{-1}]'$, which, by Lemma 2.2 (ii), is equivalent to $\beta t = (\gamma t)'$ for some $\gamma \in F$. In other words, γ is a solution of the Risch differential equation $y' + \alpha y = \beta$, or equivalently, $\beta \in \operatorname{im}(\mathcal{R}_\alpha)$, where $\mathcal{R}_\alpha$ is given in Definition 1.10. Therefore, complete reductions for $(F(t), ')$ are inherently connected to Risch differential equations.*

We are going to construct a complete reduction Φ_h for $(F(t), \mathcal{R}_h)$ for all $h \in F(t)$ under the assumption that a complete reduction for $(F, \mathcal{R}_\alpha)$ is available for every element $\alpha \in F$, where $\mathcal{R}_\alpha$ stands for the Risch operator on F associated to α.

Example 3.2. *Let $F = C$ and $c \in C$. The Risch operator $\mathcal{R}_c : C \to C$ is given by $y \mapsto y' + cy$ for all $y \in C$. Then $\mathcal{R}_c(y) = cy$. It follows that $\operatorname{im}(\mathcal{R}_c) = C$ if $c \neq 0$, and $\operatorname{im}(\mathcal{R}_c) = \{0\}$ if $c = 0$. The complete reduction for $(C, \mathcal{R}_c)$ is the zero map if $c \neq 0$, and is the identity map if $c = 0$.*

Similar to the way of solving Risch equations in [10, Theorem 6.1.1], it suffices to focus on Risch operators associated to normalized elements defined below (see also [11, Section 3.2]).

Definition 3.3. *An element h of $F(t)$ is said to be* t-normalized *if, for all $i \in \mathbb{Z}$,*

$$\gcd\left(\operatorname{num}(h) - i \operatorname{den}(h)', \operatorname{den}(h)\right) = 1$$

The gcd-condition is equivalent to the requirement that no nonzero residue of h is an integer by [10, Theorem 4.4.3]. All elements of F are t-normalized.

This normalization will simplify our construction in three aspects:

(i) Reduce the problem of constructing complete reductions for $(F(t), \mathcal{R}_h)$ with $h \in F(t)$ to that for $(F(t), \mathcal{R}_h)$ with t-normalized $h \in F(t)$ (see Proposition 3.4).

(ii) Further transform the problem in $F(t)$ into an equivalent one on $F\langle t \rangle$ (see Proposition 3.6).

(iii) The Risch operator associated to a t-normalized element h is injective if $h \in F(t) \setminus F$, t is regular, and t is either primitive or hyperexponential over F (see Lemma 4.3).

An element of $F(t)$ is said to be *weakly t-normalized* if it does not have any positive integer residue (see [19] and [10, Definition 6.1.1]). Note that (i) and (ii) remain valid for weakly t-normalized elements, while (iii) relies on the assumption that h has no residue in $\mathbb{Z}^\times$.

For every element $h \in F(t)$, there exists a pair $(\xi, \eta) \in F(t) \times F(t)^\times$ such that ξ is t-normalized, and $h = \xi + \eta'/\eta$. We call (ξ, η) a *normal form* of h. Normal forms are not unique. Among all such normal forms, the canonical one can be computed by Algorithm GKS in [11, Section 3.2].

Proposition 3.4. *Let $h \in F(t)$ with a normal form (ξ, η). Then the following assertions hold.*

(i) $\mathcal{R}_h = \eta^{-1} \circ \mathcal{R}_\xi \circ \eta$, where η is understood as the map with $f \mapsto \eta f$ for all $f \in F(t)$.

(ii) Assume further that Φ_ξ is a complete reduction for $(F(t), \mathcal{R}_\xi)$. Then $\Phi_h := \eta^{-1} \circ \Phi_\xi \circ \eta$ is a complete reduction for $(F(t), \mathcal{R}_h)$.

(iii) For all $f \in F(t)$, $\left(\eta^{-1} g, \eta^{-1} r\right)$ is an R-pair of f with respect to Φ_h if (g, r) is an R-pair of ηf with respect to Φ_ξ.

Proof. (i) For all $f \in F(t)$, $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \eta^{-1}\left((\eta f)' + \xi(\eta f)\right)$. Then $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \mathcal{R}_h(f)$ by $h = \xi + \eta'/\eta$ and a straightforward calculation.

(ii) Since Φ_ξ is an idempotent, so is Φ_h. Hence, $\operatorname{im}(\Phi_h)$ is a complement of its kernel in $F(t)$. It suffices to verify that $\ker(\Phi_h) = \operatorname{im}(\mathcal{R}_h)$. By (i), $\operatorname{im}(\mathcal{R}_h) = \eta^{-1} \cdot \operatorname{im}(\mathcal{R}_\xi)$. By $\Phi_h = \eta^{-1} \circ \Phi_\xi \circ \eta$, we have $\ker(\Phi_h) = \eta^{-1} \cdot \ker(\Phi_\xi)$. Thus, $\ker(\Phi_h) = \operatorname{im}(\mathcal{R}_h)$ by $\ker(\Phi_\xi) = \operatorname{im}(\mathcal{R}_\xi)$.

(iii) Let (g, r) be an R-pair of ηf with respect to Φ_ξ. Then $\eta f = \mathcal{R}_\xi(g) + r$ and $r \in \operatorname{im}(\Phi_\xi)$. It follows that $f = \mathcal{R}_h(\eta^{-1} g) + \eta^{-1} r$, which, together with $\eta^{-1} r \in \operatorname{im}(\Phi_h)$, implies that $\left(\eta^{-1} g, \eta^{-1} r\right)$ is an R-pair of f with respect to Φ_h. □

In the rest of this section, we let $h \in F(t)$ be t-normalized, $a = \operatorname{num}(h)$ and $b = \operatorname{den}(h)$. Set

$$S_{t,h} := \{s \in S_t \mid \gcd(\operatorname{den}(s), b) = 1\}, \tag{4}$$

which is also a C-subspace in $F(t)$. For all $f \in F(t)$, Algorithm GKSR in [11, Section 3.4] computes a triple $(g, r, s) \in F(t) \times F\langle t \rangle \times S_{t,h}$ such that

$$f = \mathcal{R}_h(g) + \frac{r}{b} + s. \tag{5}$$

Moreover, s in the above equality is unique. Therefore,

$$F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot F\langle t \rangle\right) \oplus S_{t,h}. \tag{6}$$

This direct sum generalizes the splitting $F(t) = (F(t)' + F\langle t\rangle) \oplus S_t$ induced by Algorithm HermiteReduce in [10, Section 5.6]. It remains to find a C-subspace $W \subset F\langle t\rangle$ with

$$\operatorname{im}(\mathcal{R}_h) + \left(b^{-1} \cdot F\langle t\rangle\right) = \operatorname{im}(\mathcal{R}_h) \oplus \left(b^{-1} \cdot W\right).$$

In the special case $F(t) = C(x)$, such a subspace can be constructed via a polynomial reduction map in [4, Section 4.1]. This motivates us to define the notion of companion operators.

Definition 3.5. *The map*

$$\begin{array}{cccc} \mathcal{P}_h : & F\langle t\rangle & \to & F\langle t\rangle \\ & r & \mapsto & br' + ar. \end{array}$$

is called the companion operator *of* $\mathcal{R}_h$*.*

The following proposition allows us to reduce the problem of constructing a complete reduction for $(F(t), \mathcal{R}_h)$ to that for $(F\langle t\rangle, \mathcal{P}_h)$.

Proposition 3.6. *If* W *is a complement of* $\operatorname{im}(\mathcal{P}_h)$ *in* $F\langle t\rangle$*, then*

$$F(t) = \operatorname{im}(\mathcal{R}_h) \oplus \left(b^{-1} \cdot W \oplus S_{t,h}\right). \tag{7}$$

Proof. By (6), $F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot \operatorname{im}(\mathcal{P}_h) + b^{-1} \cdot W\right) \oplus S_{t,h}$. Then

$$F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot W\right) \oplus S_{t,h}$$

by $b^{-1} \cdot \operatorname{im}(\mathcal{P}_h) \subset \operatorname{im}(\mathcal{R}_h)$. It remains to show that $\operatorname{im}(\mathcal{R}_h) \cap \left(b^{-1} \cdot W\right) = \{0\}$. Assume that $v \in \operatorname{im}(\mathcal{R}_h) \cap \left(b^{-1} \cdot W\right)$. Then $v = w/b \in \operatorname{im}(\mathcal{R}_h)$ for some $w \in W$, and thus there exists $g \in F(t)$ such that $bg' + ag = w$.

Since h is t-normalized, we have $g \in F\langle t\rangle$ by either [10, Theorem 6.1.2] or [11, Theorem 3.12 (iii)]. Hence, $w \in \operatorname{im}(\mathcal{P}_h)$. By $\operatorname{im}(\mathcal{P}_h) \cap W = \{0\}$, w is equal to 0, and so is v. □

Remark 3.7. *With the notation from the above proposition, we further let* $\Phi_{h,W}$ *be the projection from* $F(t)$ *to* $(b^{-1} \cdot W \oplus S_{t,h})$ *with respect to* (7)*, and* $\Psi_{h,W}$ *be the projection from* $F\langle t\rangle$ *to* W *with respect to* $F\langle t\rangle = \operatorname{im}(\mathcal{P}_h) \oplus W$*. Then* $\Phi_{h,W}$ *and* $\Psi_{h,W}$ *are complete reductions for* $(F(t), \mathcal{R}_h)$ *and* $(F\langle t\rangle, \mathcal{P}_h)$*, respectively.*

Assume that one can compute R-pairs with respect to $\Psi_{h,W}$*. Then R-pairs with respect to* $\Phi_{h,W}$ *can be derived as follows. Let* $f \in F(t)$*.*

1. *Find* $(g, r, s) \in F(t) \times F\langle t\rangle \times S_{t,h}$ *satisfying* (5) *by Algorithm* GKSR *in [11, Section 3.4].*
2. *Compute an R-pair* $(u, v) \in F\langle t\rangle \times W$ *of* r *with respect to* $\Psi_{h,W}$*.*

Then the pair $(g + u,\ b^{-1}v + s)$ *is an R-pair of* f *with respect to* $\Phi_{h,W}$*. This follows from the fact* $b^{-1}v + s \in \left(b^{-1} \cdot W\right) + S_{t,h}$ *and the calculation:*

$$\begin{aligned} f &= \mathcal{R}_h(g) + b^{-1}r + s && \textit{(by (5))} \\ &= \mathcal{R}_h(g) + b^{-1}\left(\mathcal{P}_h(u) + v\right) + s && \textit{(by } r = \mathcal{P}_h(u) + v\textit{)} \\ &= \mathcal{R}_h(g) + \mathcal{R}_h(u) + b^{-1}v + s && \textit{(by } \mathcal{R}_h(u) = b^{-1}\mathcal{P}_h(u)\textit{)} \\ &= \mathcal{R}_h(g + u) + b^{-1}v + s. \end{aligned}$$

Remark 3.8. *If* $h = 0$*, then* $b = 1$*, and* (7) *in Proposition 3.6 simplifies to*

$$F(t) = F(t)' \oplus (W \oplus S_t),$$

which leads to a complete reduction for $(F(t),\ ')$*.*

4 Induction hypothesis and notational convention

With the problem of constructing complete reductions now reduced to the setting of differential subrings, we proceed to formalize the induction framework that will drive our construction.

Let $n \in \mathbb{N}$ and $C(t_1, \ldots, t_{n-1}, t_n)$ be a transcendental Liouvillian extension of C throughout this section. For inductive purposes, we set $F := C(t_1, \ldots, t_{n-1})$ and $t := t_n$. By [10, equation (5.1)], the differential ring $F\langle t \rangle$ takes the form $F[t]$ when t is primitive, and $F[t, t^{-1}]$ when t is hyperexponential. This dichotomy will structure our inductive arguments in Sections 5 and 6.

With the field setting fully specified, we now state the induction hypothesis that will be used throughout the remainder of our inductive construction.

Hypothesis 4.1. *Assume that there is a map that assigns to each element α of F a complete reduction Φ_α for $(F, \mathcal{R}_\alpha)$, and that there is an algorithm for computing an R-pair for every element of F with respect to Φ_α.*

By the hypothesis, $\operatorname{im}(\mathcal{R}_\alpha) \cap \operatorname{im}(\Phi_\alpha) = \{0\}$ for all $\alpha \in F$, and every element $\beta \in F$ has an R-pair $(\gamma, \Phi_\alpha(\beta)) \in F^2$ with respect to Φ_α, that is, $\beta = \mathcal{R}_\alpha(\gamma) + \Phi_\alpha(\beta)$.

The base case of our induction is given by the following example.

Example 4.2. *Let $F = C$. The map in Hypothesis 4.1 assigns to zero the identity operator and to every element of $C^\times$ the zero operator by Example 3.2.*

For every $h \in F(t)$, Propositions 3.4 and 3.6 convert the problem of constructing complete reductions for $(F(t), \mathcal{R}_h)$ to that for $(F\langle t \rangle, \mathcal{P}_h)$ with an additional property that h is t-normalized.

Let $h \in F(t)$ be t-normalized. In the next two sections, we will construct a complete reduction for $(F\langle t \rangle, \mathcal{P}_h)$ in three steps:

1. Construct a subspace $A_h \subset F\langle t \rangle$ such that $F\langle t \rangle = \operatorname{im}(\mathcal{P}_h) + A_h$ by (10), (26) and (27).

2. Find an echelon sequence E_h of $\operatorname{im}(\mathcal{P}_h) \cap A_h$ by Propositions 5.15, 6.12 and Lemma 2.13.

3. Determine a complement W of $\operatorname{im}(\mathcal{P}_h) \cap A_h$ in A_h by E_h and Lemma 2.9.

Once these steps are completed, we obtain the direct sum decomposition

$$F\langle t \rangle = \operatorname{im}(\mathcal{P}_h) \oplus W,$$

and the projection from $F\langle t \rangle$ to W gives the desired complete reduction. The above three steps have been carried out in [21, Section 3] when t is primitive and $h = 0$. However, additional technical obstacles emerge when $h \neq 0$ or t is hyperexponential.

The notion of auxiliary subspaces comes from our attempt to simplify the coefficients of elements in $F\langle t \rangle$ modulo $\operatorname{im}(\mathcal{P}_h)$. Such a simplification relies on an elaborate application of integration by parts in the case $h = 0$ (see [14, 22]).

A basis of $\operatorname{im}(\mathcal{P}_h) \cap A_h$ is constructed based on two algorithms presented in [32, §4.3.1]: one is to recognize logarithmic derivatives in F (see [10, §5.12]); the other is to find parametric logarithmic derivatives in F (see [10, §7.3] and [32, Chapter 4]). Note that these two algorithms are well developed for transcendental Liouvillian extensions, while they cannot be guaranteed to work for general differential fields. Although such a basis may not be finite when t is primitive, its regular shape described in Proposition 5.15 allows us to find an echelon sequence of the intersection, which induces a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F\langle t \rangle$ by Lemma 2.9, and leads to an algorithm for computing R-pairs by Lemma 2.13.

We present a lemma allowing us to transform a C-basis of a subspace in $F\langle t \rangle$ to a C-basis of its image under $\mathcal{P}_h$.

Lemma 4.3. *If $h \in F(t) \setminus F$ is t-normalized, then both $\mathcal{R}_h$ and $\mathcal{P}_h$ are injective.*

Proof. Suppose that $u \in \ker(\mathcal{R}_h)^\times$. Then $h = -u'/u$ by Definition 1.10. The logarithmic derivative identity implies that h is a $\mathbb{Z}$-linear combination of logarithmic derivatives of polynomials in $F[t]$. Since h is t-normalized, no logarithmic derivative of any element in N_t appears in this $\mathbb{Z}$-linear combination. Since t is either primitive or hyperexponential over F, we derive that $h \in F$, a contradiction. Thus, $\mathcal{R}_h$ is injective. Consequently, $\mathcal{P}_h$ is injective, because it is equal to $\mathrm{den}(h)\mathcal{R}_h$ restricted to $F\langle t\rangle$. $\square$

As the final preparation for our induction, we make a notational convention that will be valid throughout the next two sections.

Convention 4.4. *(i) Let $C(t_1, \ldots, t_n)$ be a transcendental Liouvillian extension of C. Set*

$$F := C(t_1, \ldots, t_{n-1}) \quad \text{and} \quad t := t_n.$$

(ii) Assume that t is primitive over F in Section 5, and hyperexponential over F in Section 6.

(iii) h is a t-normalized element of $F(t)$.

(iv) Set $a := \mathrm{num}(h)$, $b := \mathrm{den}(h)$ and $m := \mathrm{Deg}(h)$. Write

$$a = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0 \quad \text{and} \quad b = b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0,$$

where $a_i, b_i \in F$ and $b_m = 1$ whenever $b_m \neq 0$.

5 The primitive case

By Convention 4.4 (ii), t is primitive throughout this section. Then $F\langle t\rangle = F[t]$. The coefficients a_m, b_m, a_{m-1} and b_{m-1} in Convention 4.4 (iv) will play a significant role in reducing polynomials. Note that both a_{m-1} and b_{m-1} are zero if $m = 0$.

To describe the leading terms of elements in $\mathrm{im}(\mathcal{P}_h)$, we define a C-linear operator

$$\begin{array}{rccc} \mathcal{L}: & F & \to & F \\ & \alpha & \mapsto & b_m\alpha' + a_m\alpha. \end{array} \tag{8}$$

Then, for an element $f \in F[t]$ with $d = \deg(f)$ and $f_d = \mathrm{lc}(f)$, we have

$$\mathcal{P}_h(f) \equiv \mathcal{L}(f_d) t^{m+d} \mod F[t]_{<m+d} \tag{9}$$

by $\mathcal{P}_h(f) \equiv (b_m f_d') t^{m+d} + (a_m f_d) t^{m+d} \mod F[t]_{<m+d}$.

Remark 5.1. *By Convention 4.4 and (8), the following two assertions hold.*

(i) If $\nu_\infty(h) < 0$, then $a_m \neq 0$ and $b_m = 0$. Consequently, $\mathcal{L}$ is injective.

(ii) If $\nu_\infty(h) \geq 0$, then $b_m = 1$ and $\mathcal{L} = \mathcal{R}_{a_m}$, which is the Risch operator on F associated to a_m.

The next lemma will be used frequently for constructing echelon sequences in this section.

Lemma 5.2. *If $\lambda \in \ker(\mathcal{L})^\times$, then $\Phi_{a_m}(\lambda t') \neq 0$, where Φ_{a_m} is the complete reduction for $(F, \mathcal{R}_{a_m})$ given in Hypothesis 4.1.*

Proof. Since $\ker(\mathcal{L}) \neq \{0\}$, we have $\nu_\infty(h) \geq 0$ and then $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1. It follows that $\lambda \in \ker(\mathcal{R}_{a_m})^\times$. Thus, $\lambda' + a_m\lambda = 0$. Suppose that $\Phi_{a_m}(\lambda t') = 0$. Then $\lambda t' \in \mathrm{im}(\mathcal{R}_{a_m})$. Hence, there exists $\mu \in F$ such that $\mu' + a_m\mu = \lambda t'$. The two equalities involving a_m imply that $t' = (\mu/\lambda)'$, a contradiction to the regularity of t. $\square$

The rest of this section has three parts. We first define the auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ and develop a corresponding reduction from $F[t]$ to U_h modulo $\mathrm{im}(\mathcal{P}_h)$ in Section 5.1. Section 5.2 presents a criterion to determine whether $\mathrm{im}(\mathcal{P}_h) \cap U_h$ is trivial. An echelon sequence of this intersection, together with a complete reduction for $(F(t), \mathcal{R}_h)$, is constructed in Section 5.3.

5.1 Auxiliary subspaces

The idea of auxiliary subspaces originates from the attempt to decompose an element $f \in F[t]$ as $g' + r$, where $' = \mathcal{R}_0$ is the derivation on $F[t]$, $g, r \in F[t]$, $\deg(r) \leq \deg(f)$, and all coefficients of r lie in $\mathrm{im}(\Phi_0)$. Note that Φ_0 is the complete reduction for $(F, ')$ given in Hypothesis 4.1. Such a decomposition is always possible by [21, Lemma 3.1].

We generalize this idea from $'$ to $\mathcal{R}_h$, where h is given in Convention 4.4 (iii) and (iv).

To this end, we let $f \in F[t]$, $d = \deg(f)$ and $f_d = \mathrm{lc}(f)$, and assume that $d \geq m$. Then

$$f \equiv \begin{cases} \mathcal{P}_h\left(a_m^{-1} f_d t^{d-m}\right) \mod F[t]_{<d} & \text{if } \nu_\infty(h) < 0, \\ \\ \mathcal{P}_h\left(\alpha t^{d-m}\right) + \beta t^d \mod F[t]_{<d} & \text{if } \nu_\infty(h) \geq 0, \end{cases} \tag{10}$$

where $(\alpha, \beta) \in F^2$ is an R-pair of f_d with respect to Φ_{a_m} given in Hypothesis 4.1.

The first congruence holds by expanding $\mathcal{P}_h\left(a_m^{-1} f_d t^{d-m}\right)$ directly modulo $F[t]_{<d}$ and noticing that $b_m = 0$ due to $\nu_\infty(h) < 0$. To show the second congruence, we note that $f_d = \mathcal{R}_{a_m}(\alpha) + \beta$ by Definition 1.5. Then $f_d = \mathcal{L}(\alpha) + \beta$ by Remark 5.1 (ii). So $f \equiv \mathcal{L}(\alpha)t^d + \beta t^d \mod F[t]_{<d}$. On the other hand, $\mathcal{P}_h\left(\alpha t^{d-m}\right) \equiv \mathcal{L}(\alpha)t^d \mod F[t]_{<d}$ by (9). Combining the last two congruences, we prove the second congruence in (10).

Iteratively applying the two congruences in (10) to elements of $F[t]_{<d}$, we arrive at the following definition.

Definition 5.3. *The* auxiliary subspace associated to $(F[t], \mathcal{P}_h)$ *is defined as*

$$U_h := \begin{cases} F[t]_{<m} & \text{if } \nu_\infty(h) < 0, \\ \\ F[t]_{<m} + \sum_{i \in \mathbb{N}_0} \mathrm{im}(\Phi_{a_m}) \cdot t^{m+i} & \text{if } \nu_\infty(h) \geq 0. \end{cases}$$

The set U_h is a C-subspace of $F[t]$ because both $F[t]_{<m}$ and $\mathrm{im}(\Phi_{a_m})$ are C-subspaces of $F[t]$. Indeed, U_h is C-linearly isomorphic to $F[t]_{<m} + \mathrm{im}\left(\Phi_{a_m}\right) \otimes_C C[t]$ when $\nu_\infty(h) \geq 0$. The above definition generalizes the notion of auxiliary subspaces in [21] for the case where $m = 0$ and $a_m = 0$.

Example 5.4. *Let $F = C$, $t = x$ and $' = d/dx$. By Convention 4.4,*

	a	b	m	a_m	b_m	$\nu_\infty(h)$
$h = 0$	0	1	0	0	1	∞
$h = x$	x	1	1	1	0	-1

It follows from Example 4.2 that the auxiliary subspaces associated to $(F[t], \mathcal{P}_0)$ and $(F[t], \mathcal{P}_x)$ are $C[x]$ and C, respectively.

Proposition 5.5. *Let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and $f \in F[t]^\times$ with $d = \deg(f)$. Then there exists a pair $(g, r) \in F[t] \times U_h$ with $\deg(g) \leq d - m$ and $\deg(r) \leq d$ such that $f = \mathcal{P}_h(g) + r$. Consequently, $F[t] = \operatorname{im}(\mathcal{P}_h) + U_h$.*

Proof. If $d < m$, then $f \in U_h$. So it suffices to set $g := 0$ and $r := f$. Assume that $d \geq m$, and that the conclusion holds for polynomials of degrees lower than d. Let $f_d = \operatorname{lc}(f)$.

If $\nu_\infty(h) < 0$, then $f \equiv \mathcal{P}_h\left(a_m^{-1} f_d t^{d-m}\right) \mod F[t]_{<d}$ by (10). Applying the induction hypothesis to $F[t]_{<d}$, we see that the proposition holds.

Otherwise, $\nu_\infty(h) \geq 0$. Let (α, β) be an R-pair of f_d with respect to Φ_{a_m}. Then

$$f \equiv \mathcal{P}_h(\alpha t^{d-m}) \mod U_h + F[t]_{<d}$$

by the second congruence in (10) and $\beta \in \operatorname{im}(\Phi_{a_m})$.

It follows from the induction hypothesis and $F[t]_{<m} \subset U_h$ that $f \equiv \mathcal{P}_h(g) \mod U_h$ for some $g \in F[t]$ with $\deg(g) \leq d - m$. Furthermore, $\deg(\mathcal{P}_h(g)) \leq d$ by (9). The conclusion holds by setting $r := f - \mathcal{P}_h(g)$. □

Definition 5.6. *Let f and (g, r) be given in the above proposition. We call (g, r) an* auxiliary pair *of f with respect to $(F[t], \mathcal{P}_h)$, or an* auxiliary pair *of f if $(F[t], \mathcal{P}_h)$ is clear from context.*

The proof of Proposition 5.5 leads to Algorithm A.1 in Section A.1. The next corollary reveals a relation between auxiliary pairs and R-pairs in the special case $h \in F$.

Corollary 5.7. *Let $h \in F$, and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then, for every element $f \in F$, an auxiliary pair of f with respect to $(F[t], \mathcal{P}_h)$ is an R-pair of f with respect to Φ_h.*

Proof. Since $f \in F$, its degree is zero. So every auxiliary pair (α, β) of f with respect to $(F[t], \mathcal{P}_h)$ belongs to F^2 by Proposition 5.5. It remains to show that (α, β) is an R-pair of f with respect to Φ_h. With Convention 4.4, we have the following table for notation:

	a	b	m	a_m	b_m	$\nu_\infty(h)$
$h \in F$	h	1	0	h	1	≥ 0

Then the auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ is $\sum_{i \in \mathbb{N}_0} \operatorname{im}(\Phi_h) \cdot t^i$ by Definition 5.3. So $\beta \in \operatorname{im}(\Phi_h)$ by $\beta \in U_h \cap F$. On the other hand, the restrictions of $\mathcal{R}_h$ and $\mathcal{P}_h$ to F are equal by Definitions 1.10 and 3.5. Therefore, $f = \mathcal{P}_h(\alpha) + \beta = \mathcal{R}_h(\alpha) + \beta$. It follows from $F = \operatorname{im}(\mathcal{R}_h) \oplus \operatorname{im}(\Phi_h)$ that $\beta = \Phi_h(f)$, that is, (α, β) is an R-pair with respect to Φ_h. □

5.2 Intersecting $\operatorname{im}(\mathcal{P}_h)$ with the associated auxiliary subspace

In this subsection, we let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$. By Proposition 5.5, $F[t] = \operatorname{im}(\mathcal{P}_h) + U_h$. Then $F[t] = \operatorname{im}(\mathcal{P}_h) \oplus U_h$ if and only if $\operatorname{im}(\mathcal{P}_h) \cap U_h = \{0\}$. By setting $I_h := \operatorname{im}(\mathcal{P}_h) \cap U_h$, we aim to derive a necessary and sufficient condition on $I_h = \{0\}$.

The following lemma describes the leading coefficient of every element in I_h.

Lemma 5.8. *If $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]$, then $\operatorname{lc}(f) \in \ker(\mathcal{L})$.*

Proof. Let $d = \deg(f)$ and $f_d = \operatorname{lc}(f)$.

If $\nu_\infty(h) < 0$, then $U_h = F[t]_{<m}$ by Definition 5.3. So the degree of $\mathcal{P}_h(f)$ is less than m. It follows from (9) that $\mathcal{L}(f_d) = 0$.

Assume that $\nu_\infty(h) \geq 0$. Then $\mathcal{L}(f_d) \in \operatorname{im}(\Phi_{a_m})$ by (9), $\mathcal{P}_h(f) \in U_h$ and Definition 5.3. Note that $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1 (ii). Since $\operatorname{im}(\mathcal{R}_{a_m}) \cap \operatorname{im}(\Phi_{a_m})$ is equal to $\{0\}$, so is $\operatorname{im}(\mathcal{L}) \cap \operatorname{im}(\Phi_{a_m})$. Accordingly, $\mathcal{L}(f_d) = 0$. □

The next proposition is a criterion for $I_h = \{0\}$.

Proposition 5.9. *$I_h = \{0\}$ if and only if $\ker(\mathcal{L}) = \{0\}$. In particular, $I_h = \{0\}$ if $\nu_\infty(h) < 0$.*

Proof. Assume that $\ker(\mathcal{L}) = \{0\}$. Let $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]$. By Lemma 5.8, $\operatorname{lc}(f)$ is equal to 0, and so is f. Thus, $I_h = \{0\}$.

Conversely, assume that $I_h = \{0\}$. Let $\lambda \in \ker(\mathcal{L})$. We need to show $\lambda = 0$. By (9), we have that $\mathcal{P}_h(\lambda) \in F[t]_{<m}$. Then $\mathcal{P}_h(\lambda) \in U_h$ by Definition 5.3. Consequently, $\mathcal{P}_h(\lambda) = 0$ by $I_h = \{0\}$.

If $h \in F(t) \setminus F$, then $\lambda = 0$ by Lemma 4.3.

Assume that $h \in F$. Then $m = 0$, $a = a_m = h$ and $b = b_m = 1$ by Convention 4.4. So $\mathcal{P}_h(\lambda t) = \lambda t'$ by $\lambda \in \ker(\mathcal{L})$. Since $\lambda t' \in F$, it has an auxiliary pair of the form $(\mu, \Phi_h(\lambda t'))$ for some $\mu \in F$ by Corollary 5.7, where Φ_h is the complete reduction for $(F, \mathcal{R}_h)$ given in Hypothesis 4.1. Thus $\mathcal{P}_h(\lambda t) = \lambda t' = \mathcal{P}_h(\mu) + \Phi_h(\lambda t')$. Consequently, $\mathcal{P}_h(\lambda t - \mu) = \Phi_h(\lambda t')$, which, together with $\Phi_h(\lambda t') \in U_h$, implies that $\Phi_h(\lambda t') \in I_h$. We conclude from $I_h = \{0\}$ that $\Phi_h(\lambda t') = 0$. Then $\lambda = 0$ by Lemma 5.2.

If $\nu_\infty(h) < 0$, then $\ker(\mathcal{L}) = \{0\}$ by Remark 5.1 (i). So $I_h = \{0\}$. □

We distinguish whether I_h is trivial or not by types defined below.

Definition 5.10. *The* type *of I_h is defined as 0 if $I_h = \{0\}$ (i.e., $\ker(\mathcal{L}) = \{0\}$) and otherwise, as an element $\lambda \in \ker(\mathcal{L})^\times$.*

The type of I_h is unique up to a multiplicative constant in $C^\times$, because $\dim_C \ker(\mathcal{L}) \le 1$. It can be found by recognizing logarithmic derivatives in F.

Example 5.11. *Let $h = 0$. Then $m = 0$, $a_m = 0$ and $b_m = 1$ by Convention 4.4. Thus, $\mathcal{L} = {}'$ by (8). Consequently, $\ker(\mathcal{L}) = C$. The type of I_0 can be taken as 1.*

5.3 Echelon sequences and induced complements

The goal of this subsection is to construct an echelon sequence of I_h, yielding a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F[t]$. Let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and set $I_h := \operatorname{im}(\mathcal{P}_h) \cap U_h$ as before. Assume further that $I_h \neq \{0\}$ and its type is a fixed element $\lambda \in \ker(\mathcal{L})^\times$. We proceed in the following four steps:

1. Construct a C-basis B_1 of $\mathcal{P}_h^{-1}(I_h)$ in Lemma 5.14.
2. Describe a C-basis B_2 of I_h in Proposition 5.15 by mapping B_1 to I_h via $\mathcal{P}_h$ with a minor modification.
3. Convert B_2 into an echelon sequence of I_h in Corollaries 5.17 and 5.18.
4. Construct a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F[t]$ by Lemma 2.9.

In the rest of this subsection, every basis is a C-basis. So we omit the prefix "C- " for brevity.

5.3.1 Standard basis of $\mathcal{P}_h^{-1}(I_h)$

Recall the notation given in Convention 4.4. We have

$$m = \operatorname{Deg}(h), \quad \operatorname{num}(h) = \sum_{i=0}^{m} a_i t^i \quad \text{and} \quad \operatorname{den}(h) = \sum_{i=0}^{m} b_i t^i.$$

By Proposition 5.9, $\nu_\infty(h) \ge 0$. So $b_m = 1$ and $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1 (ii).

In view of Lemma 5.8 and $\dim_C(\ker(\mathcal{L})) = 1$, all elements of $\mathcal{P}_h^{-1}(I_h)^\times$ must have leading coefficient $c\lambda$ for some $c \in C^\times$. This observation motivates us to compute the leading term of $\mathcal{P}_h(\lambda t^i)$ for all $i \in \mathbb{N}_0$. By Definition 3.5 and $\mathcal{L}(\lambda) = 0$, we have that

$$\mathcal{P}_h(\lambda t^i) \equiv (i\lambda t' + b_{m-1}\lambda' + a_{m-1}\lambda) \cdot t^{m+i-1} \mod F[t]_{<m+i-1} \tag{11}$$

for all $i \in \mathbb{N}_0$. To express the coefficient $i\lambda t' + b_{m-1}\lambda' + a_{m-1}\lambda$ of t^{m+i-1} in terms of R-pairs, we introduce the following definition.

Definition 5.12. *Let Φ_{a_m} be the complete reduction for $(F, \mathcal{R}_{a_m})$ given in Hypothesis 4.1. A* first R-pair associated to *$(F[t], \mathcal{P}_h)$ is an R-pair of $\lambda t'$ with respect to Φ_{a_m}, and a* second one *is an R-pair of $b_{m-1}\lambda' + a_{m-1}\lambda$ with respect to Φ_{a_m}.*

An element may admit R-pairs with different first components. For the rest of this section, we fix the first and second associated R-pairs, and denote them by $(\tilde{\sigma}, \sigma)$ and $(\tilde{\tau}, \tau)$, respectively. Note that $(\tilde{\tau}, \tau) := (0, 0)$ if $m = 0$.

Remark 5.13. *The entry σ in the first associated R-pair $(\tilde{\sigma}, \sigma)$ is nonzero by Lemma 5.2.*

With $\lambda t' = \mathcal{R}_{a_m}(\tilde{\sigma}) + \sigma$ and $b_{m-1}\lambda' + a_{m-1}\lambda = \mathcal{R}_{a_m}(\tilde{\tau}) + \tau$, we rewrite (11) as

$$\mathcal{P}_h(\lambda t^i) \equiv \mathcal{R}_{a_m}(i\tilde{\sigma} + \tilde{\tau}) \cdot t^{m+i-1} + (i\sigma + \tau) \cdot t^{m+i+1} \mod F[t]_{<m+i-1}. \tag{12}$$

In order to construct a basis for $\mathcal{P}_h^{-1}(I_h)$, we first define a sequence $\{p_i\}_{i\in\mathbb{N}_0}$ in $F[t]$ such that $\deg(p_i) = i$, $\mathrm{lc}(p_i) = \lambda$ and $p_i \in \mathcal{P}_h^{-1}(I_h)$.

For $i = 0$, we let $p_0 = \lambda$. Then $\mathcal{P}_h(p_0) \in F[t]_{<m}$ by (9). Since $F[t]_{<m} \subset U_h$, we have $\mathcal{P}_h(p_0) \in I_h$ and, consequently, $p_0 \in \mathcal{P}_h^{-1}(I_h)$.

Assume that $i > 0$. It follows from (12) and $\mathcal{L} = \mathcal{R}_{a_m}$ that

$$\mathcal{P}_h(\lambda t^i) \equiv \mathcal{L}(i\tilde{\sigma} + \tilde{\tau}) \cdot t^{m+i-1} + (i\sigma + \tau) \cdot t^{m+i-1} \mod F[t]_{<m+i-1},$$

which, together with (9), implies that there exists a polynomial $g_i \in F[t]_{<m+i-1}$ such that

$$\mathcal{P}_h(\lambda t^i) = \mathcal{P}_h\left((i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1}\right) + (i\sigma + \tau) \cdot t^{m+i-1} + g_i.$$

Let (q_i, r_i) be the auxiliary pair of g_i obtained from Algorithm A.1. Then $\deg(q_i) < i - 1$, $\deg(r_i) < m + i - 1$ by Proposition 5.5. Moreover,

$$\mathcal{P}_h(\lambda t^i) = \mathcal{P}_h\left((i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1}\right) + (i\sigma + \tau) \cdot t^{m+i-1} + \mathcal{P}_h(q_i) + r_i.$$

Moving the images under $\mathcal{P}_h$ in the above equality to the left-hand side, we arrive at

$$\mathcal{P}_h\left(\lambda t^i - (i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1} - q_i\right) = (i\sigma + \tau) \cdot t^{m+i-1} + r_i.$$

Set

$$p_i := \lambda t^i - (i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1} - q_i. \tag{13}$$

Then $\deg(p_i) = i$ and

$$\mathcal{P}_h(p_i) = (i\sigma + \tau) \cdot t^{m+i-1} + r_i. \tag{14}$$

Since $\sigma, \tau \in \mathrm{im}(\Phi_{a_m})$ and $r_i \in U_h$, we see that $\mathcal{P}_h(p_i) \in U_h$, that is, $p_i \in \mathcal{P}_h^{-1}(I_h)$ for all $i \in \mathbb{N}$.

Lemma 5.14. *Let $p_0 = \lambda$ and p_i be defined by (13) for all $i \in \mathbb{N}$. Then $\{p_i\}_{i\in\mathbb{N}_0}$ is a basis of $\mathcal{P}_h^{-1}(I_h)$.*

Proof. We have seen that $\{p_i\}_{i\in\mathbb{N}_0} \subset \mathcal{P}_h^{-1}(I_h)$. Since $\deg(p_i) = i$ for all $i \in \mathbb{N}_0$, the set $\{p_i\}_{i\in\mathbb{N}_0}$ is C-linearly independent. Taking $f \in F[t]$ such that $\mathcal{P}_h(f) \in U_h$, we see that $\operatorname{lc}(f) \in \ker(\mathcal{L})$ by Lemma 5.8. Then $\operatorname{lc}(f) = c\lambda$ for some $c \in C$ by $\lambda \in \ker(\mathcal{L})^\times$ and $\dim_C(\ker(\mathcal{L})) = 1$.

Let $d = \deg(f)$. Then $f - cp_d \in \mathcal{P}_h^{-1}(I_h)$ with $\deg(f - cp_d) < d$. Thus, f is a C-linear combination of $p_d, \ldots, p_0$ by a straightforward induction on d. $\square$

We call $\{p_i\}_{i\in\mathbb{N}_0}$ the *standard basis* of $\mathcal{P}_h^{-1}(I_h)$, where $p_0 = \lambda$ and p_i is given in (13) for $i \in \mathbb{N}$. This basis is unique, because the type of I_h and the two associated R-pairs are all fixed, and every auxiliary pair (q_i, r_i) is computed by Algorithm A.1.

Let us summarize the notation introduced so far in the following table.

	Type	1st R-pair	2nd R-pair	Standard basis
Expression	λ	$(\tilde{\sigma}, \sigma)$	$(\tilde{\tau}, \tau)$	$p_0 = \lambda$, p_i in (13), $i \in \mathbb{N}$
Property	$\lambda \in \ker(\mathcal{L})^\times$	$\lambda t' = \mathcal{R}_{a_m}(\tilde{\sigma}) + \sigma$	$b_{m-1}\lambda' + a_{m-1}\lambda$ $= \mathcal{R}_{a_m}(\tilde{\tau}) + \tau$	A basis of $\mathcal{P}_h^{-1}(I_h)$

Table 1: Notational summary in Section 5.3.1

5.3.2 Basis of I_h

We transform the standard basis derived above into a basis of I_h using the companion operator $\mathcal{P}_h$ as formulated below.

Proposition 5.15. *With the notation given in Table 1, we further let r_i be given in* (14).

(i) If $h \in F$, then $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ is a basis of I_h, and $\mathcal{P}_h(p_i) = i\sigma t^{i-1} + r_i$ with degree $i - 1$.

(ii) If $h \in F(t) \setminus F$, then $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}_0}$ is a basis of I_h, $\mathcal{P}_h(p_0) \in F[t]^\times_{<m}$, and, for all $i \in \mathbb{N}$,

$$\mathcal{P}_h(p_i) = (i\sigma + \tau)t^{m+i-1} + r_i.$$

Proof. (i) Since $h \in F$, the restriction of $\mathcal{P}_h$ to F is equal to $\mathcal{L}$. Then $\mathcal{P}_h(p_0) = 0$ by $p_0 = \lambda$. Hence, $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ spans I_h over C by Lemma 5.14. Moreover, $h \in F$ implies that $m = 0$ and $(\tilde{\tau}, \tau) = (0, 0)$. It follows from (14) that $\mathcal{P}_h(p_i) = i\sigma t^{i-1} + r_i$. This image is of degree $i - 1$ by Remark 5.13 and $\deg(r_i) < i - 1$. Thus, $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ is C-linearly independent, and then it is a basis of I_h.

(ii) Since $h \in F(t) \setminus F$, the companion operator $\mathcal{P}_h$ is injective by Lemma 4.3, which, together with Lemma 5.14, implies that $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}_0}$ is a basis of I_h. Note that $\mathcal{P}_h(p_0) = \mathcal{P}_h(\lambda)$, which is a nonzero polynomial of degree less than m by $\lambda \in \ker(\mathcal{L})$ and (9). The rest follows by (14). $\square$

The following example rediscovers the complete reduction in Example 1.2.

Example 5.16. *Let $F = C$ and $t = x$ with $x' = 1$. Then $F(t) = C(x)$. Set $h = 0$. Then the auxiliary subspace U_0 is equal to $C[x]$ by Example 5.4. The calculations carried out for constructing the standard basis of $\mathcal{P}_0^{-1}(I_0)$ lead to the following table:*

Type λ	*1st R-pair* $(\tilde{\sigma}, \sigma)$	*2nd R-pair* $(\tilde{\tau}, \tau)$	p_i, $i \in \mathbb{N}_0$	$\mathcal{P}_0(p_i)$, $i \in \mathbb{N}$
1	$(0, 1)$	$(0, 0)$	x^i	ix^{i-1}

Then $I_0 = C[x]$ by Proposition 5.15 (i). Consequently, $C[x] = \operatorname{im}(\mathcal{P}_0)$. Hence, $\{0\}$ is the only complement of $\operatorname{im}(\mathcal{P}_0)$ in $C[x]$. By Proposition 3.6, $C(x) = C(x)' \oplus S_x$.

5.3.3 Echelon sequences

We construct echelon sequences of I_h according to whether $h \in F$ or not. We refer to Definition 2.10 for the definition of echelon sequences.

Let Θ be an effective basis of F (see Definition 2.12). Then

$$\Omega := \{\vartheta t^i \mid \vartheta \in \Theta, i \in \mathbb{N}_0\} \tag{15}$$

is an effective basis of $F[t]$ by [21, Remark 2.7].

Recall that $(\tilde{\sigma}, \sigma)$ and $(\tilde{\tau}, \tau)$ are the first and second R-pairs associated to $(F[t], \mathcal{P}_h)$, respectively. In the rest of this section, we fix an element $\vartheta_\sigma \in \Theta$ with $\vartheta_\sigma^*(\sigma) \neq 0$. By Remark 5.13, such an element exists. It will be used for describing most of the pivots in an echelon sequence.

The next corollary is immediate from Proposition 5.15 (i).

Corollary 5.17. *With the notation in Table 1, we further let $h \in F$. Then*

$$\left\{ \left(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{i-1}\right) \right\}_{i \in \mathbb{N}}$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω.

When $h \in F(t) \setminus F$, constructing an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω requires a careful and tedious case study, because the elements of $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}_0}$ in Proposition 5.15 (ii) do not necessarily have distinct degrees. A detailed case study is carried out in [4, Section 4.1] for the special case in which $F = C$, $t = x$ and $' = d/dx$.

Corollary 5.18. *With the notation given in Table 1, we further let $h \in F(t) \setminus F$ with degree m. Assume that d_0 and l_0 are the degree and leading coefficient of $\mathcal{P}_h(p_0)$, respectively, and set ϑ_0 to be an element of Θ such that $\vartheta_0^*(l_0) \neq 0$.*

(i) If $\vartheta_\sigma^*(i\sigma + \tau) \neq 0$ for all $i \in \mathbb{N}$, then

$$\left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^m\right), \left(p_2, \mathcal{P}_h(p_2), \vartheta_\sigma t^{m+1}\right), \ldots, \left(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω, which is illustrated as

preimages	*basis elements*	*degrees of basis elements*	*pivots*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m+i-1$	$\vartheta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\vartheta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 *with* $d_0 < m$	$\vartheta_0 t^{d_0}$

where the length of a line segment corresponds to the degree of an element in the basis, and so does it in the next two pictures.

(ii) If $\vartheta_\sigma^*(j\sigma + \tau) = 0$ for some $j \in \mathbb{N}$ but $j\sigma + \tau \neq 0$, then there exists an element $\vartheta \in \Theta$ with $\operatorname{lc}(\mathcal{P}_h(p_j)) \notin \ker(\vartheta^*)$ such that

$$\left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \vartheta_\sigma t^{m+j-2}\right),$$

$$\left(p_j, \mathcal{P}_h(p_j), \vartheta t^{m+j-1}\right), \left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \vartheta_\sigma t^{m+j}\right), \ldots, \left(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω, which is illustrated as

preimages	*basis elements*	*degrees of basis elements*	*pivots*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m+i-1$	$\vartheta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$m+j$	$\vartheta_\sigma t^{m+j}$
p_j	$\mathcal{P}_h(p_j)$	$m+j-1$	ϑt^{m+j-1}
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$m+j-2$	$\vartheta_\sigma t^{m+j-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\vartheta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 *with* $d_0 < m$	$\vartheta_0 t^{d_0}$

(iii) If $j\sigma + \tau = 0$ for some $j \in \mathbb{N}$, then there exist $q \in F[t]^\times$, $d \in \mathbb{N}_0$, and $\vartheta \in \Theta$ with $\operatorname{lc}(\mathcal{P}_h(q)) \notin \ker(\vartheta^*)$ such that

$$\left(q, \mathcal{P}_h(q), \vartheta t^d\right), \left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \vartheta_\sigma t^{m+j-2}\right),$$
$$\left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \vartheta_\sigma t^{m+j}\right), \ldots, \left(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω, which is illustrated as

preimages	*basis elements*	*degrees of basis elements*	*pivots*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m+i-1$	$\vartheta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$m+j$	$\vartheta_\sigma t^{m+j}$
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$m+j-2$	$\vartheta_\sigma t^{m+j-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\vartheta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 *with* $d_0 < m$	$\vartheta_0 t^{d_0}$
q	$\mathcal{P}_h(q)$	d *with* $d < m+j-1$	ϑt^d

Proof. (i) The conclusion is immediate from Proposition 5.15 (ii).

(ii) Since $j\sigma + \tau \neq 0$, we see that $\mathcal{P}_h(p_j)$ is of degree $m+j-1$ by Proposition 5.15 (ii). Note that j is the unique integer such that $\vartheta_\sigma^*(j\sigma+\tau) = 0$. So $\vartheta_\sigma^*(i\sigma+\tau) \neq 0$ for all $i \in \mathbb{N}$ with $i \neq j$. Hence, the degree of $\mathcal{P}_h(p_i)$ is $m+i-1$ for all $i \in \mathbb{N}$ with $i \neq j$. Let ϑ be an element of Θ such that the leading coefficient of $\mathcal{P}_h(p_j)$ does not belong to $\ker(\vartheta^*)$. We see that (ii) holds.

(iii) Since $j\sigma + \tau = 0$, the degree of $\mathcal{P}_h(p_j)$ is less than $m+j-1$ by Proposition 5.15 (ii). Let U be the subspace spanned by $\mathcal{P}_h(p_0)$, $\mathcal{P}_h(p_1)$, ..., $\mathcal{P}_h(p_{j-1})$ over C. Then

$$\left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \vartheta_\sigma t^{m+j-2}\right)$$

is an echelon sequence of U with respect to $(F[t], \mathcal{P}_h)$ and Ω.

Set $\omega_0 := \vartheta_0 t^{d_0}$, $\omega_1 := \vartheta_\sigma t^m$, $\ldots$, $\omega_{j-1} := \vartheta_\sigma t^{m+j-2}$. It follows from Lemma 2.13 that there exists a polynomial $r \in F[t]$ such that $\mathcal{P}_h(p_j) - \mathcal{P}_h(r) \in \cap_{k=0}^{j-1} \ker(\omega_k^*)$ and $\mathcal{P}_h(r) \in U$.

Set $q := p_j - r$. Then $\mathcal{P}_h(q)$ lies in both I_h and $\cap_{k=0}^{j-1} \ker(\omega_k^*)$. Hence,

$$\mathcal{P}_h(q), \mathcal{P}_h(p_0), \mathcal{P}_h(p_1), \ldots, \mathcal{P}_h(p_{j-1}), \mathcal{P}_h(p_{j+1}), \mathcal{P}_h(p_{j+2}), \ldots$$

form a basis of I_h, because $\{\mathcal{P}_h(p_0), \mathcal{P}_h(p_1), \ldots, \mathcal{P}_h(p_{j-1}), \mathcal{P}_h(p_j), \mathcal{P}_h(p_{j+1}), \mathcal{P}_h(p_{j+2}), \ldots\}$ is a basis of I_h. Let d be the degree of $\mathcal{P}_h(q)$, and choose an element $\vartheta \in \Theta$ such that the leading coefficient of $\mathcal{P}_h(q)$ does not belong to $\ker(\vartheta^*)$. We conclude that (iii) holds. $\square$

5.3.4 Induced complements

Let $E := \{(f_i, \mathcal{P}_h(f_i), \omega_i)\}_{i \in \mathbb{N}}$ be an echelon sequence of I_h given in Corollaries 5.17 or 5.18. It follows from Lemma 2.9 that the subspace

$$U_h \cap \Big(\bigcap_{i \in \mathbb{N}} \ker(\omega_i^*) \Big) \tag{16}$$

is a complement of $\mathrm{im}(\mathcal{P}_h)$ in $F[t]$, which is called the *complement of* $\mathrm{im}(\mathcal{P}_h)$ *induced by* E.

Every complement of $\mathrm{im}(\mathcal{P}_h)$ in $F[t]$ is finite-dimensional over C if $F = C$ by Example 5.16 and [4, Lemma 8]. However, such a subspace is of infinite dimension if $\dim_C(F) = \infty$.

Example 5.19. *Let $F = C(x)$, $t = \log(x)$ and d/dx be the derivation on $F(t)$. Let*

$$h = \frac{2x^2 - 2t}{xt^2 + x},$$

which is t-normalized by a direct verification.

We construct an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω in (15). By Convention 4.4, we have the following table:

a	b	m	a_m	b_m	a_{m-1}	b_{m-1}
$-2x^{-1}t + 2x$	$t^2 + 1$	2	0	1	$-2x^{-1}$	0

Since $a_m = 0$, we may take the Hermite-Ostrogradsky reduction to be the complete reduction Φ_0 for $(F, ')$ in Hypothesis 4.1. The auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ is equal to

$$F[t]_{<2} + \sum_{i \in \mathbb{N}_0} S_x \cdot t^{i+2} \tag{17}$$

by Definition 5.3 and $\mathrm{im}(\Phi_0) = S_x$.

Note that $\mathcal{L}$ in (8) is equal to d/dx by $a_2 = 0$ and $b_2 = 1$. Then we can set the type λ of I_h to be 1. By Algorithm A.1, the first associated R-pair is $(\tilde{\sigma}, \sigma) = (0, x^{-1})$ and the second R-pair is $(\tilde{\tau}, \tau) = (0, -2x^{-1})$. Since $\sigma = x^{-1}$, we choose ϑ_σ given before Corollary 5.17 to be x^{-1}. Corollary 5.18 (iii) is needed to construct an echelon sequence E, because $2\sigma + \tau = 0$.

We first compute p_0 and p_1 in the standard basis $\{p_i\}_{i \in \mathbb{N}_0}$ of $\mathcal{P}_h^{-1}(I_h)$. Since the type λ is equal to 1, we have $p_0 = 1$, and then $\mathcal{P}_h(p_0) = -2x^{-1}t + 2x$. Hence, $d_0 = 1$. Choose $\vartheta_0 = x^{-1}$. Then the second member of E is

$$(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}) = \left(1, -2x^{-1}t + 2x, x^{-1}t\right). \tag{18}$$

Note that $\vartheta_0 = \vartheta_\sigma$ is just a coincidence. By (13) and $\tilde{\sigma} = \tilde{\tau} = 0$, we see that $p_1 = t$. Thus, the third member of E is

$$(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^2) = \left(t, -x^{-1}t^2 + 2xt + x^{-1}, x^{-1}t^2\right). \tag{19}$$

Since $2\sigma + \tau = 0$, we compute the first member of E by p_2. By (13) and Algorithm A.1, we find $p_2 = t^2 - x^2$. Then $\mathcal{P}_h(p_2) = (2x + 2x^{-1})t - 2x^3 - 2x$. Eliminating $x^{-1}t$ from $\mathcal{P}_h(p_2)$ by $\mathcal{P}_h(p_0)$ yields $(q, \mathcal{P}_h(q)) = \left(t^2 - x^2 + 1, 2xt - 2x^3\right)$. Choose $\vartheta = x$. Then the first member is

$$(q,\, \mathcal{P}_h(q),\, \vartheta t) = \left(t^2 - x^2 + 1,\, 2xt - 2x^3,\, xt\right). \tag{20}$$

For all integers $k > 3$, the kth member of E can be found by (13), (14) and Algorithm A.1. Its pivot is equal to $x^{-1}t^k$ by Corollary 5.18 (iii). We summarize the results in the following table:

λ	$(\tilde{\sigma}, \sigma)$	$(\tilde{\tau}, \tau)$	ϑ_σ	j	$(q, \mathcal{P}_h(q), \vartheta t)$	$(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0})$	$(p_1, \mathcal{P}_h(p_1), \vartheta_\sigma t^2)$
1	$\left(0, x^{-1}\right)$	$\left(0, -2x^{-1}\right)$	x^{-1}	2	*r.h.s. of* (20)	*r.h.s. of* (18)	*r.h.s. of* (19)

where j stands for the positive integer with $j\sigma + \tau = 0$.

By (16) and (17), the complement W of $\operatorname{im}(\mathcal{P}_h)$ induced by E is equal to

$$\Big(C(x)[t]_{<2} + \sum_{i\in\mathbb{N}_0} S_x \cdot t^{i+2}\Big) \cap \Big(\bigcap_{i\in\mathbb{N}} \ker(\omega_i^*)\Big),$$

where $\omega_1 = xt$, $\omega_2 = x^{-1}t$, $\omega_3 = x^{-1}t^2$ as listed in the table above, and $\omega_k = x^{-1}t^k$ for $k > 3$ following our conclusion about the pivot of the kth member in E. Any element $r \in W$ can be written in the form

$$r = r_0 + r_1 t + r_2 t^2 + r_3 t^3 + r_4 t^4 + \cdots + r_k t^k$$

with $r_i \in C(x)$ satisfying the following conditions:

- *in the irreducible partial fraction decomposition of r_1, the polynomial part has no term of degree 1, and the proper part contains no term of the form c/x for any $c \in C^\times$;*
- *r_3 is an arbitrary element of S_x;*
- *$r_2, r_4, \ldots, r_k$ all belong to S_x and have no pole at $x = 0$.*

We have made three choices for pivots in the above example. Together with (13) and Algorithm A.1, these choices uniquely determine an echelon sequence of I_h. To systematically study such constructions, we introduce the notion of initial sequences in Definition A.2 of the appendix. Every initial sequence is finite and carries all the information required for the unique construction of an echelon sequence. For instance, the final table in the above example encodes an initial sequence. Moreover, Algorithm A.4 computes initial sequences, and Algorithm A.5 generates arbitrary many members of the echelon sequence determined by a given initial sequence.

With Remark 1.6, we are ready to present the main result of this section.

Theorem 5.20. *Assume that Hypothesis 4.1 holds and t is primitive. Then there exists an algorithm that, for every t-normalized element h in $F(t)$, constructs a complete reduction $\widetilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.*

Proof. Recall that U_h is the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and that λ is the type of I_h. If $\lambda = 0$, then we can set $W := U_h$ by Proposition 5.5.

Otherwise, let $E = \{(q_i, Q_i, \omega_i)\}_{i\in\mathbb{N}}$ be the echelon sequence of I_h obtained from Corollary 5.17 or Corollary 5.18. Set W to be the complement of $\operatorname{im}(\mathcal{P}_h)$ induced by E. Then $F[t] = \operatorname{im}(\mathcal{P}_h) \oplus W$. It follows from Proposition 3.6 that

$$F(t) = \operatorname{im}(\mathcal{R}_h) \oplus \left(b^{-1} \cdot W \oplus S_{t,h}\right), \tag{21}$$

where $b = \operatorname{den}(h)$ by Convention 4.4, and $S_{t,h}$ is given in (4). Therefore, the projection from $F(t)$ to $\left(b^{-1} \cdot W \oplus S_{t,h}\right)$ with respect to (21) gives a complete reduction $\widetilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.

It remains to describe an algorithm for computing R-pairs with respect to $\widetilde{\Phi}_h$. Let $\widetilde{\Psi}_h$ be the projection from $F[t]$ to W with respect to $F[t] = \operatorname{im}(\mathcal{P}_h) \oplus W$. By Remark 3.7, it suffices to develop an algorithm for computing R-pairs with respect to $\widetilde{\Psi}_h$.

Take any $p \in F[t]$. An auxiliary pair (q, r) of p can be found by Algorithm A.1. If $\lambda = 0$, then (q, r) is an R-pair with respect to $\widetilde{\Psi}_h$. Otherwise, one can compute $\tilde{q} \in F[t]$ and $\tilde{r} \in I_h$ such that $\tilde{r} = \mathcal{P}_h(\tilde{q})$ and $r - \tilde{r} \in \cap_{i\in\mathbb{N}} \ker(\omega_i^*)$ by Lemma 2.13. On the other hand, $r - \tilde{r} \in U_h$ by $r, \tilde{r} \in U_h$. Thus, $r - \tilde{r} \in W$ by (16). Consequently, $(q + \tilde{q},\, r - \tilde{r})$ is an R-pair of p with respect to $\widetilde{\Psi}_h$. ☐

The next corollary relates the complete reduction $\widetilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$ from the preceding theorem to the complete reduction Φ_h for $(F, \mathcal{R}_h)$ specified in Hypothesis 4.1 whenever $h \in F$. It is useful for elementary integration in Section 8.

Corollary 5.21. *Let $h \in F$, Φ_h be the complete reduction for $(F,\, \mathcal{R}_h)$ in Hypothesis 4.1, and $\widetilde{\Phi}_h$ be the complete reduction for $(F(t),\, \mathcal{R}_h)$ in Theorem 5.20. The following two assertions hold.*

(i) For all $f \in F$, $\widetilde{\Phi}_h(f) = \Phi_h(f)$ if $I_h = \{0\}$, and, otherwise, there exists a constant $c \in C$ such that $\widetilde{\Phi}_h(f) = \Phi_h(f) + c\Phi_h(\lambda t')$, where λ is the type of I_h.

(ii) $\operatorname{im}(\widetilde{\Phi}_0) \subset \operatorname{im}(\Phi_0) \oplus (t \cdot F[t]) \oplus S_t$.

Proof. (i) Since $h \in F$, we see that $b = 1$ by Convention 4.4. Then $S_{t,h}$ in (21) is equal to S_t. Hence, equation (21) becomes

$$F(t) = \operatorname{im}(\mathcal{R}_h) \oplus (W \oplus S_t)\,. \tag{22}$$

Moreover, $\widetilde{\Phi}_h$ is the projection from $F(t)$ to $W \oplus S_t$ with respect to this direct sum.

For every $f \in F$, it follows from $h \in F$ and Corollary 5.7 that

$$f = \mathcal{R}_h(\alpha) + \Phi_h(f) \tag{23}$$

for some $\alpha \in F$, and that $(\alpha, \Phi_h(f))$ is an auxiliary pair of f with respect to $(F[t], \mathcal{P}_h)$.

If $I_h = \{0\}$, then W in (22) is equal to U_h. Since $\mathcal{R}_h(\alpha) \in \operatorname{im}(\mathcal{R}_h)$ and $\Phi_h(f) \in U_h$, we conclude by (23) that $\widetilde{\Phi}_h(f) = \Phi_h(f)$.

Assume that $I_h \neq \{0\}$. Then W in (22) is the complement induced by the echelon sequence E of I_h given by Corollary 5.17. It follows from $\Phi_h(f) \in U_h$ and (23) that $\widetilde{\Phi}_h(f)$ is the projection of $\Phi_h(f)$ on W with respect to $U_h = I_h \oplus W$.

Let $(\tilde{\sigma}, \sigma)$ be the first R-pair associated to $(F[t], \mathcal{P}_h)$. Then $\sigma = \Phi_h(\lambda t')$ by Definition 5.12. Note that the first member of E is $(\lambda t - \tilde{\sigma}, \Phi_h(\lambda t'), \vartheta_\sigma)$. Set $c = -\vartheta_\sigma^*\left(\Phi_h(\lambda t')\right)^{-1} \cdot \vartheta_\sigma^*\left(\Phi_h(f)\right)$. We find that $\Phi_h(f) + c\Phi_h(\lambda t')$ belongs to W, because other pivots in E are of positive degree in t. Therefore, $\widetilde{\Phi}_h(f) = \Phi_h(f) + c\Phi_h(\lambda t')$.

(ii) By Definition 5.3, the auxiliary subspace associated to $(F[t], \mathcal{P}_0)$ is

$$U_0 = \sum_{i\in\mathbb{N}_0} \operatorname{im}(\Phi_0) \cdot t^i = \operatorname{im}(\Phi_0) + \sum_{i\in\mathbb{N}} \operatorname{im}(\Phi_0) \cdot t^i,$$

which is contained in $\operatorname{im}(\Phi_0) + (t \cdot F[t])$. Since $W \subset U_0$, we have $W \subset \operatorname{im}(\Phi_0) + (t \cdot F[t])$. Thus, $\operatorname{im}(\widetilde{\Phi}_0) \subset \operatorname{im}(\Phi_0) + (t \cdot F[t]) + S_t$. The sum of the three subspaces is evidently direct. ☐

6 The hyperexponential case

This section is organized in the same way as in Section 5. In terms of content, there are two substantive differences. First, the definition of auxiliary subspaces becomes more involved due to the presence of negative powers. Second, the construction of echelon sequences turns out to be considerably simpler, since the intersection of $\operatorname{im}(\mathcal{P}_h)$ with the auxiliary subspace associated to $(F[t,t^{-1}], \mathcal{P}_h)$ is of dimension at most two over C (see Proposition 6.12).

The results in this section generalize those in [24, Chapter 4] for the exponential case and those in [11, Section 4.2] for the rationally hyperexponential case (see [11, Definition 4.2]).

We keep the assumptions and notation in Convention 4.4. Then t is hyperexponential and $F\langle t\rangle = F[t,t^{-1}]$. The tail coefficients a_0 and b_0 will play a significant role in reducing negative powers of t.

To describe the head and tail terms of an element in $\operatorname{im}(\mathcal{P}_h)$, we define two C-linear operators as follows. For every $d \in \mathbb{Z}$, let

$$\begin{array}{rccc} \mathcal{H}_d: & F & \to & F \\ & \alpha & \mapsto & b_m\alpha' + \left(a_m + d b_m \dfrac{t'}{t}\right)\alpha \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathcal{T}_d: & F & \to & F \\ & \alpha & \mapsto & b_0\alpha' + \left(a_0 + d b_0 \dfrac{t'}{t}\right)\alpha. \end{array} \tag{24}$$

For every $f \in F[t,t^{-1}]^\times$ with $k = \operatorname{hdeg}(f)$, $f_k = \operatorname{hc}(f)$, $l = \operatorname{tdeg}(f)$ and $f_l = \operatorname{tc}(f)$, we have

$$\mathcal{P}_h(f) = \mathcal{H}_k(f_k)t^{m+k} + \cdots + \mathcal{T}_l(f_l)t^l \tag{25}$$

by Lemma 2.2 (ii). It follows that $\mathcal{P}_h(f^+) \in F[t]$ and $\mathcal{P}_h(f^-) \in F[t]_{<m} + t^{-1}F[t^{-1}]$.

Remark 6.1. *By Convention 4.4 and* (24)*, the following assertions hold for all $k, l \in \mathbb{Z}$.*

(i) If $\nu_\infty(h) < 0$, then $a_m \neq 0$, $b_m = 0$ and $\ker(\mathcal{H}_k) = \{0\}$.

(ii) If $\nu_\infty(h) \geq 0$, then $b_m = 1$ and $\mathcal{H}_k = \mathcal{R}_{\lambda_k}$, where $\lambda_k = a_m + k\dfrac{t'}{t}$.

(iii) If $\nu_t(h) < 0$, then $a_0 \neq 0$, $b_0 = 0$, and $\ker(\mathcal{T}_l) = \{0\}$.

(iv) If $\nu_t(h) \geq 0$, then $b_0 \neq 0$ and $\mathcal{T}_l = b_0\mathcal{R}_{\mu_l}$, where $\mu_l = \dfrac{a_0}{b_0} + l\dfrac{t'}{t}$.

The λ_k's and μ_l's defined above will be used throughout the rest of this section.

In this section, we will mainly prove conclusions concerning tail degrees. For the head-degree case, the reasoning is analogous to the arguments developed in the previous section. Actually, the head-degree case is easier to handle than the tail-degree case because either $b_m = 1$ or $b_m = 0$.

6.1 Auxiliary subspaces

As in Section 5.1, we aim to reduce all coefficients of elements in $F[t,t^{-1}]$ modulo $\operatorname{im}(\mathcal{P}_h)$.

Let $f \in F[t,t^{-1}]$ with $k = \operatorname{hdeg}(f)$, $f_k = \operatorname{hc}(f)$, $l = \operatorname{tdeg}(f)$ and $f_l = \operatorname{tc}(f)$. Assume further that $k \geq m$ and $l < 0$. By arguments analogous to those at the beginning of Section 5.1, we obtain the following four congruences.

$$f^+ \equiv \begin{cases} \mathcal{P}_h\left(a_m^{-1}f_k t^{k-m}\right) \mod F[t]_{<k} & \text{if } \nu_\infty(h) < 0, \\ \mathcal{P}_h(\alpha t^{k-m}) + \beta t^k \mod F[t]_{<k} & \text{if } \nu_\infty(h) \geq 0, \end{cases} \tag{26}$$

where $(\alpha, \beta) \in F^2$ is an R-pair of f_k with respect to $\Phi_{\lambda_{k-m}}$ given in Hypothesis 4.1, and

$$f^- \equiv \begin{cases} \mathcal{P}_h\left(a_0^{-1} f_l t^l\right) \mod F[t]_{<m} + \left(t^{-1} \cdot F[t^{-1}]\right)_{>l} & \text{if } \nu_t(h) < 0, \\ \\ \mathcal{P}_h(\alpha t^l) + (b_0\beta) t^l \mod F[t]_{<m} + \left(t^{-1} \cdot F[t^{-1}]\right)_{>l} & \text{if } \nu_t(h) \geq 0, \end{cases} \tag{27}$$

where $(\alpha, \beta) \in F^2$ is an R-pair of $b_0^{-1} f_l$ with respect to Φ_{μ_l} given in Hypothesis 4.1. Note that $\nu_t(h) < 0$ implies $a_0 \neq 0$ by Remark 6.1 (iii) and $\nu_t(h) \geq 0$ implies $b_0 \neq 0$ by Remark 6.1 (iv).

The two congruences in (26) can be verified by the same reasoning used to establish (10). The first congruence in (27) holds by (25) and Remark 6.1 (iii). To show the second congruence in (27), we note that $b_0^{-1} f_l = \mathcal{R}_{\mu_l}(\alpha) + \beta$, which, together with $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$ given in Remark 6.1 (iv), implies that $f_l = \mathcal{T}_l(\alpha) + b_0\beta$. So $f^- \equiv \mathcal{T}_l(\alpha) t^l + (b_0\beta)\, t^l \mod F[t]_{<m} + \left(t^{-1} \cdot F[t^{-1}]\right)_{>l}$. The congruence follows from (25).

The above deduction motivates the notion of auxiliary subspaces below.

Definition 6.2. *The auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$ is defined as*

$$V_h = V_h^+ + F[t]_{<m} + V_h^-,$$

where

$$V_h^+ = \begin{cases} \{0\} & \text{if } \nu_\infty(h) < 0, \\ \\ \displaystyle\sum_{i \geq m} \operatorname{im}\left(\Phi_{\lambda_{i-m}}\right) \cdot t^i & \text{if } \nu_\infty(h) \geq 0, \end{cases} \quad \text{and } V_h^- = \begin{cases} \{0\} & \text{if } \nu_t(h) < 0, \\ \\ \displaystyle\sum_{j<0} b_0 \cdot \operatorname{im}\left(\Phi_{\mu_j}\right) \cdot t^j & \text{if } \nu_t(h) \geq 0. \end{cases}$$

Example 6.3. *Let $F = C$ and $t'/t = 1$, that is, t models $\exp(x)$. Let $h = 0$. Convention 4.4 and Remark 6.1 lead to the following table:*

m	a_m	a_0	b_m	b_0	λ_k, $k \geq 0$	μ_l, $l < 0$
0	0	0	1	1	k	l

From Example 4.2, we obtain $V_h^+ = C$ and $V_h^- = \{0\}$. So C is the auxiliary subspace V_h associated to $(F[t, t^{-1}], \mathcal{P}_h)$.

Analogous to Proposition 5.5 and Definition 5.6, we have

Proposition 6.4. *Let $f \in F[t, t^{-1}]$, $k = \operatorname{hdeg}(f)$ and $l = \operatorname{tdeg}(f)$. Then*

(i) $f^+ = \mathcal{P}_h(p) + r$ for some $p \in F[t]$ with $\operatorname{hdeg}(p) \leq k - m$ and $r \in V_h$,

(ii) $f^- = \mathcal{P}_h(q) + s$ for some $q \in t^{-1} \cdot F[t^{-1}]$ with $\operatorname{tdeg}(q) \geq l$ and $s \in V_h$.

Consequently, $F[t, t^{-1}] = \operatorname{im}(\mathcal{P}_h) + V_h$.

Proof. (i) The proof is analogous to that of Proposition 5.5 (i).

(ii) If $f^- = 0$, then $l \geq 0$. It suffices to set $q := 0$ and $s := 0$. Assume that $l < 0$ and the conclusion holds for all elements of $\left(t^{-1} \cdot F[t^{-1}]\right)_{>l}$.

If $\nu_t(h) < 0$, then $f^- \equiv \mathcal{P}_h\left(a_0^{-1} f_l t^l\right) + g \mod F[t]_{<m}$ for some $g \in t^{-1} F[t^{-1}]_{>l}$ by the first congruence in (27). The conclusion holds by an application of the induction hypothesis to g.

Otherwise, $\nu_t(h) \geq 0$. Let (α, β) be an R-pair of $b_0^{-1} f_l$ with respect to Φ_{μ_l}. It follows from the second congruence in (27), $F[t]_{<m} \subset V_h$ and $(b_0\beta) \cdot t^l \in V_h^-$ that $f^- \equiv \mathcal{P}_h(\alpha t^l) + g \mod V_h$ for some $g \in \left(t^{-1} \cdot F[t^{-1}]\right)_{>l}$. The conclusion holds by the induction hypothesis.

By (i) and (ii), $f = \mathcal{P}_h(p + q) + r + s$ and $r + s \in V_h$. Hence, $F[t, t^{-1}] = \operatorname{im}(\mathcal{P}_h) + V_h$. □

Definition 6.5. *Let f, p, q, r and s be as in the above proposition. We call $(p+q, r+s)$ an* auxiliary pair *of f with respect to $(F[t,t^{-1}], \mathcal{P}_h)$ or an* auxiliary pair *of f if $(F[t,t^{-1}], \mathcal{P}_h)$ is clear from context.*

Based on the congruences in (26), (27), and the proof of Proposition 6.4, an algorithm is developed for computing auxiliary pairs (see Algorithm A.7 in the appendix).

Corollary 6.6. *Let $h \in F$, and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then, for every element $f \in F$, an auxiliary pair of f with respect to $\left(F[t,t^{-1}], \mathcal{P}_h\right)$ is an R-pair of f with respect to Φ_h.*

Proof. Note that λ_0 defined in Remark 6.1 (ii) is equal to h under the assumption $h \in F$. The rest is similar to the proof of Corollary 5.7. □

6.2 Intersecting $\mathrm{im}(\mathcal{P}_h)$ with the associated auxiliary subspace

In this subsection, V_h stands for the auxiliary subspace associated to $(F[t,t^{-1}], \mathcal{P}_h)$, and J_h denotes $\mathrm{im}(\mathcal{P}_h) \cap V_h$. We will derive a necessary and sufficient condition on $J_h = \{0\}$.

The next two lemmas connect J_h with the kernels of $\mathcal{H}_d$ and $\mathcal{T}_d$ in (24). We only prove the second assertion in each lemma, and the first assertion can be shown likewise.

Lemma 6.7. *Let $f \in F[t,t^{-1}]^\times$, $k = \mathrm{hdeg}(f)$ and $l = \mathrm{tdeg}(f)$. Assume that $\mathcal{P}_h(f) \in V_h$. Then*

(i) $\mathrm{hc}(f) \in \ker(\mathcal{H}_k)$ *if $k \geq 0$, and*

(ii) $\mathrm{tc}(f) \in \ker(\mathcal{T}_l)$ *if $l < 0$.*

Proof. (ii) Assume that $l < 0$ and let $f_l = \mathrm{tc}(f)$. Suppose that $\mathcal{T}_l(f_l) \neq 0$. Then $\mathcal{T}_l(f_l)t^l \in V_h^-$ by $\mathcal{P}_h(f) \in V_h$ and $l < 0$. In particular, $V_h^- \neq \{0\}$. So $\nu_t(h) \geq 0$ and $\mathcal{T}_l(f_l) \in b_0 \cdot \mathrm{im}(\Phi_{\mu_l})$ by Definition 6.2. By Remark 6.1 (iv), $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$. Thus, $\mathcal{R}_{\mu_l}(f_l) \in \mathrm{im}(\Phi_{\mu_l})$. Since $\mathrm{im}(\mathcal{R}_{\mu_l}) \cap \mathrm{im}(\Phi_{\mu_l})$ is equal to $\{0\}$, we derive that $\mathcal{T}_l(f_l) = 0$, a contradiction. □

Lemma 6.8. *(i) If $\ker(\mathcal{H}_k) \neq \{0\}$ for some $k \in \mathbb{N}_0$, then for all $\sigma \in \ker(\mathcal{H}_k)^\times$, there exists $p \in F[t]$ with $\mathrm{hdeg}(p) = k$ and $\mathrm{hc}(p) = \sigma$ such that $\mathcal{P}_h(p) \in V_h$. Moreover, for every $f \in F[t,t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists a constant $c \in C$ such that $f - cp \in t^{-1} \cdot F[t^{-1}]$.*

(ii) If $\ker(\mathcal{T}_l) \neq \{0\}$ for some $l \in \mathbb{N}^-$, then for all $\tau \in \ker(\mathcal{T}_l)^\times$, there exists $q \in t^{-1} \cdot F[t^{-1}]$ with $\mathrm{tdeg}(q) = l$ and $\mathrm{tc}(q) = \tau$ such that $\mathcal{P}_h(q) \in V_h$. Moreover, for every $f \in F[t,t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists a constant $c \in C$ such that $f - cq \in F[t]$.

Proof. (ii) By Remark 6.1 (iii), $\nu_t(h) \geq 0$. By (25) and $\tau \in \ker(\mathcal{T}_l)$, the image $\mathcal{P}_h(\tau t^l)$ belongs to $F[t]_{<m} + \left(t^{-1} \cdot F[t^{-1}]\right)_{>l}$. Let (g, r) be an auxiliary pair of $\mathcal{P}_h\left(\tau t^l\right)$. Then

$$\mathcal{P}_h\left(\tau t^l\right) = \mathcal{P}_h(g) + r, \quad g \in \left(t^{-1} \cdot F[t^{-1}]\right)_{>l}, \quad \text{and} \quad r \in V_h$$

by Proposition 6.4 (ii). Set $q := \tau t^l - g$. Then $\mathrm{tdeg}(q) = l$, $\mathrm{tc}(q) = \tau$ and $\mathcal{P}_h(q) \in V_h$.

Assume that $f \in F[t,t^{-1}]$ with $d = \mathrm{tdeg}(f)$ and $\mathcal{P}_h(f) \in V_h$. If $d \geq 0$, then we set $c := 0$. Otherwise, $\mathrm{tc}(f) \in \ker(\mathcal{T}_d)$ by Lemma 6.7 (ii). It follows from [10, Lemma 6.2.2] that $d = l$. Since $\dim_C(\ker(\mathcal{T}_l)) = 1$, there exists a constant $c \in C$ such that $\mathrm{tc}(f) = c\tau$. Then $\mathcal{P}_h(f - cq) \in V_h$ and $\mathrm{tdeg}(f - cq) > l$. So $\mathrm{tdeg}(f - cq) \geq 0$ by Lemma 6.7 (ii) and [10, Lemma 6.2.2]. □

We now present a criterion for $J_h = \{0\}$.

Proposition 6.9. *$J_h = \{0\}$ if and only if one of the following two conditions holds.*

(i) $h \in F$,

(ii) $\ker(\mathcal{H}_k) = \{0\}$ *for all* $k \in \mathbb{N}_0$ *and* $\ker(\mathcal{T}_l) = \{0\}$ *for all* $l \in \mathbb{N}^-$.

In particular, $J_h = \{0\}$ *if* $\nu_\infty(h) < 0$ *and* $\nu_t(h) < 0$.

Proof. Assume that $J_h = \{0\}$ and $h \notin F$. Suppose that $\ker(\mathcal{H}_k)$ is nontrivial for some $k \in \mathbb{N}_0$. Then there exists a polynomial $p \in F[t]^\times$ such that $\mathcal{P}_h(p) \in V_h$ by Lemma 6.8 (i). Moreover, $\mathcal{P}_h(p) \neq 0$ by Lemma 4.3. So J_h is nontrivial, a contradiction. The same contradiction is reached by Lemma 6.8 (ii) and Lemma 4.3 if $\ker(\mathcal{T}_l)$ is nontrivial for some $l \in \mathbb{N}^-$.

To show the converse, we let $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$. It suffices to show that $\mathcal{P}_h(f) = 0$.

First, we consider the case in which $h \in F$. Then $m = 0$, $b_0 = 1$, $\nu_\infty(h) \geq 0$ and $\nu_t(h) \geq 0$. It follows from Definition 6.2 that

$$V_h = \sum_{i \in \mathbb{N}_0} \operatorname{im}(\Phi_{\lambda_i}) \cdot t^i + \sum_{j \in \mathbb{N}^-} \operatorname{im}(\Phi_{\mu_j}) \cdot t^j,$$

where $\lambda_i = it'/t + a_0$ and $\mu_j = jt'/t + a_0$ by Remark 6.1 (ii) and (iv).

Write $f = \sum_{i=l}^k f_i t^i$ with $f_i \in F$. Since $\mathcal{P}_h(f) = f' + a_0 f$, we have

$$\mathcal{P}_h(f) = \sum_{i=0}^{k} \mathcal{R}_{\lambda_i}(f_i) t^i + \sum_{j=l}^{-1} \mathcal{R}_{\mu_j}(f_j) t^j$$

by Lemma 2.2 (ii). Hence, $\mathcal{R}_{\lambda_i}(f_i) \in \operatorname{im}(\Phi_{\lambda_i})$ for all $i \in \mathbb{N}_0$ and $\mathcal{R}_{\mu_j}(f_j) \in \operatorname{im}(\Phi_{\mu_j})$ for all $j \in \mathbb{N}^-$. Thus, $\mathcal{P}_h(f) = 0$ by $\operatorname{im}(\mathcal{R}_{\lambda_i}) \cap \operatorname{im}(\Phi_{\lambda_i}) = \{0\}$ and $\operatorname{im}(\mathcal{R}_{\mu_j}) \cap \operatorname{im}(\Phi_{\mu_j}) = \{0\}$.

Next, we assume that $h \notin F$, $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$. Suppose that $\mathcal{P}_h(f) \neq 0$. If $k = \operatorname{hdeg}(f) \geq 0$, then $f_k \in \ker(\mathcal{H}_k)^\times$ by Lemma 6.7 (i), a contradiction. So $f^+ = 0$. Thus, we may further assume that $\operatorname{tdeg}(f) = l < 0$. It follows from Lemma 6.7 (ii) that $f_l \in \ker(\mathcal{T}_l)^\times$. We have reached another contradiction.

In particular, if $\nu_\infty(h) < 0$ and $\nu_t(h) < 0$, then both $\mathcal{H}_k$ and $\mathcal{T}_l$ are injective by Remark 6.1 (i) and (iii), respectively. So $J_h = \{0\}$ by the second condition. ☐

We introduce the notion of types to characterize J_h.

Definition 6.10. *The* type *of* J_h *is defined to be*

(i) 0 *if either* $h \in F$, *or* $\ker(\mathcal{H}_k) = \{0\}$ *for all* $k \in \mathbb{N}_0$ *and* $\ker(\mathcal{T}_l) = \{0\}$ *for all* $l \in \mathbb{N}^-$,

(ii) (k, σ) *if* $h \notin F$, $\sigma \in \ker(\mathcal{H}_k)^\times$ *for some* $k \in \mathbb{N}_0$ *and* $\ker(\mathcal{T}_l) = \{0\}$ *for all* $l \in \mathbb{N}^-$,

(iii) (l, τ) *if* $h \notin F$, $\ker(\mathcal{H}_k) = \{0\}$ *for all* $k \in \mathbb{N}_0$ *and* $\tau \in \ker(\mathcal{T}_l)^\times$ *for some* $l \in \mathbb{N}^-$,

(iv) $(k, \sigma), (l, \tau)$ *if* $h \notin F$, $\sigma \in \ker(\mathcal{H}_k)^\times$ *for some* $k \in \mathbb{N}_0$ *and* $\tau \in \ker(\mathcal{T}_l)^\times$ *for some* $l \in \mathbb{N}^-$.

The type of J_h can be computed by an algorithm for solving the parametric logarithmic derivative problem in F. The type is zero if and only if $J_h = \{0\}$ by Proposition 6.9. If it is nonzero, then k, l are unique by [10, Lemma 6.2.2], and σ, τ are unique up to multiplication by a constant in $C^\times$. The type is subsequently fixed once and for all.

Example 6.11. *Let* $F = C$, $t = \exp(x)$ *and* $h = -t^2/(t^2+t+1)$. *Convention 4.4 and Remark 6.1 lead to the following table:*

m	a_m	a_0	b_m	b_0	$\mathcal{H}_k(z),\ k \geq 0$	$\mathcal{T}_l(z),\ l < 0$
2	-1	0	1	1	$(k-1)z$	lz

Hence, the type of J_h *is equal to* $(1, 1)$.

6.3 Echelon sequences and induced complements

In contrast to the primitive setting, we have $\dim_C(J_h) \le 2$, as stated in the following proposition.

Proposition 6.12. *(i)* $\dim_C(J_h) = 1$ *if the type of* J_h *is equal to* $(i, \varkappa)$ *for some* $i \in \mathbb{Z}$.

(ii) $\dim_C(J_h) = 2$ *if the type of* J_h *is equal to* $(k, \sigma), (l, \tau)$ *for some* $k \in \mathbb{N}_0$ *and* $l \in \mathbb{N}^-$.

Proof. Since the type of J_h is nonzero in both (i) and (ii), we see that $h \in F(t) \setminus F$ by Proposition 6.9. Consequently, $\mathcal{P}_h$ is injective by Lemma 4.3.

(i) Let us consider the case in which $i < 0$. By Lemma 6.8 (ii), there exists an element $q \in t^{-1} \cdot F[t^{-1}]$ with $\operatorname{tdeg}(q) = i$ such that $\mathcal{P}_h(q) \in V_h$ and for every $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, $g := f - cq \in F[t]$ for some $c \in C$. Then $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f), \mathcal{P}_h(q) \in V_h$. Suppose that $g \neq 0$. Let $k = \operatorname{hdeg}(g)$. Then $k \ge 0$ by $g \in F[t]$. It follows from Lemma 6.7 (i) that $\ker(\mathcal{H}_k) \neq \{0\}$, a contradiction to the type of J_h. So $g = 0$. Consequently, $\mathcal{P}_h(f) = c\mathcal{P}_h(q)$. Note that $\mathcal{P}_h(q) \neq 0$ by the injectivity of $\mathcal{P}_h$. So $\dim_C(J_h) = 1$.

Likewise, one shows that $\dim_C(J_h) = 1$ if $i \ge 0$.

(ii) By Lemma 6.8, there exist two elements $p \in F[t]$ with $\operatorname{hdeg}(p) = k$ and $q \in t^{-1} \cdot F[t^{-1}]$ with $\operatorname{tdeg}(q) = l$ such that $\mathcal{P}_h(p), \mathcal{P}_h(q) \in V_h$. Let $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$. Then $g := f - c_p p \in t^{-1} \cdot F[t^{-1}]$ for some $c_p \in C$ by Lemma 6.8 (i). Moreover, $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f), \mathcal{P}_h(p) \in V_h$. It follows from Lemma 6.8 (ii) that $g - c_q q \in F[t]$ for some $c_q \in C$. Then $g - c_q q = 0$ because both g and q belong to $t^{-1} \cdot F[t^{-1}]$. Thus, $f = c_p p + c_q q$. Consequently, $\mathcal{P}_h(f) = c_p \mathcal{P}_h(p) + c_q \mathcal{P}_h(q)$. Since $p \in F[t]$ and $q \in t^{-1}F[t^{-1}]$, $\mathcal{P}_h(p)$ and $\mathcal{P}_h(q)$ are C-linearly independent by the injectivity of $\mathcal{P}_h$. We conclude that $\dim_C(J_h) = 2$. □

As in Section 5, we let Θ be an effective basis of F and set $\Omega := \{\vartheta t^i \mid \vartheta \in \Theta, i \in \mathbb{Z}\}$, which is an effective basis of $F[t, t^{-1}]$. We are going to construct an echelon sequence of J_h. For the sake of completeness, the echelon sequence is set to be $\emptyset$ if the type of J_h is equal to 0.

Proposition 6.13. *Assume that* $J_h \neq \{0\}$.

(i) If the type of J_h *is equal to* $(i, \varkappa)$ *for some* $i \in \mathbb{Z}$*, then* J_h *has an echelon sequence:* $(p, \mathcal{P}_h(p), \omega)$*, and* $V_h \cap \ker(\omega^*)$ *is a complement of* $\operatorname{im}(\mathcal{P}_h)$ *in* $F[t, t^{-1}]$.

(ii) If the type of J_h *is equal to* $(k, \sigma), (l, \tau)$ *for some* $k \in \mathbb{N}_0$ *and* $l \in \mathbb{N}^-$*, then* J_h *has an echelon sequence:* $(p_1, \mathcal{P}_h(p_1), \omega_1)$, $(p_2, \mathcal{P}_h(p_2), \omega_2)$ *, and* $V_h \cap \ker(\omega_1^*) \cap \ker(\omega_2^*)$ *is a complement of* $\operatorname{im}(\mathcal{P}_h)$ *in* $F[t, t^{-1}]$.

Proof. (i) By Proposition 6.12 (i), we have $\dim_C(J_h) = 1$. Then $J_h = \operatorname{span}_C\{\mathcal{P}_h(p)\}$ for some $p \in F[t, t^{-1}]$. Let d and γ be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively. We choose an element $\vartheta \in \Theta$ such that $\vartheta^*(\gamma) \neq 0$. Then $\left(p, \mathcal{P}_h(p), \vartheta t^d\right)$ is an echelon sequence of J_h. It follows from Lemma 2.9 that $V_h \cap \ker(\omega^*)$ is a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.

(ii) By Proposition 6.12 (ii), we have $\dim_C(J_h) = 2$. Then $J_h = \operatorname{span}_C\{\mathcal{P}_h(p), \mathcal{P}_h(q)\}$ for some $p, q \in F[t, t^{-1}]$. Let d and γ be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively, and let $\vartheta_p \in \Theta$ with $\vartheta_p^*(\gamma) \neq 0$. Set $\omega_p := \vartheta_p t^d$, $c_p := \omega_p^*(\mathcal{P}_h(p))$, $c_q := \omega_p^*(\mathcal{P}_h(q))$, and $r := q - c_p^{-1} c_q p$. Then $\mathcal{P}_h(r)$ belongs to $\ker(\omega_p^*)$ and $\{\mathcal{P}_h(r), \mathcal{P}_h(p)\}$ is also a basis of J_h. Let e and ρ be the head degree and head coefficient of $\mathcal{P}_h(r)$, respectively, and choose $\vartheta_r \in \Theta$ with $\vartheta_r^*(\rho) \neq 0$. Set $\omega_r = \vartheta_r t^e$. Then $(r, \mathcal{P}_h(r), \omega_r)$, $(p, \mathcal{P}_h(p), \omega_p)$ is an echelon sequence of J_h. By Lemma 2.9, $V_h \cap \ker(\omega_r^*) \cap \ker(\omega_p^*)$ is a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$. The proof is completed by setting $p_1 := r, \omega_1 := \omega_r, p_2 := p$ and $\omega_2 := \omega_p$. □

An algorithm for constructing an echelon sequence of J_h can be developed according to the proofs of Propositions 6.12 and 6.13 (see Algorithm A.8 in the appendix).

Definition 6.14. *Let E be the echelon sequence in Proposition 6.13 (i) or (ii). The complement of* $\mathrm{im}(\mathcal{P}_h)$ *in* $F[t, t^{-1}]$ *in the same proposition is called the* complement induced by E.

In analogy to Theorem 5.20 and Corollary 5.21, we have the following.

Theorem 6.15. *Assume that Hypothesis 4.1 holds and t is hyperexponential. Then there exists an algorithm that, for every t-normalized element h of $F(t)$, constructs a complete reduction $\widetilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.*

Proof. It suffices to construct a complement W of $\mathrm{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$ by Proposition 3.6.

Recall that V_h denotes the auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$, and that J_h denotes $\mathrm{im}(\mathcal{P}_h) \cap V_h$. If the type of J_h is equal to zero, then $J_h = \{0\}$ and we simply set $W := V_h$ by Proposition 6.4. Assume that the type of J_h is equal to $(i, \varkappa)$ for some $i \in \mathbb{Z}$. By Proposition 6.13 (i), J_h has an echelon sequence $(p, \mathcal{P}_h(p), \omega)$, and we take $W := V_h \cap \ker(\omega^*)$ by Lemma 2.9. Otherwise, J_h has an echelon sequence: $(p_1, \mathcal{P}_h(p_1), \omega_1)$, $(p_2, \mathcal{P}_h(p_2), \omega_2)$ by Proposition 6.13 (ii) and we set $W := V_h \cap \ker(\omega_1^*) \cap \ker(\omega_2^*)$ by Lemma 2.9.

An algorithm for computing R-pairs with respect to $\widetilde{\Phi}_h$ can be developed along the same lines as the construction outlined in the last two paragraphs in the proof of Theorem 5.20. $\square$

Corollary 6.16. *Let $h \in F$, $\widetilde{\Phi}_h$ be the complete reduction for $(F(t), \mathcal{R}_h)$ in Theorem 6.15, and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then*

(i) $\widetilde{\Phi}_h(f) = \Phi_h(f)$ *for all* $f \in F$, *and*

(ii) $\mathrm{im}(\widetilde{\Phi}_0) \subset \mathrm{im}(\Phi_0) \oplus \left(t^{-1} \cdot F[t^{-1}]\right) \oplus (t \cdot F[t]) \oplus S_t$.

Proof. (i) By Proposition 6.9 and $h \in F$, we have $J_h = \{0\}$. The corollary follows from Corollary 6.6, via the same reasoning used in the proof of Corollary 5.21 (i) to verify $\widetilde{\Phi}_h(f) = \Phi_h(f)$ for all $f \in F$ when $I_h = \{0\}$.

(ii) The auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_0)$ is

$$\sum_{k \in \mathbb{N}} \mathrm{im}\left(\Phi_{\lambda_k}\right) \cdot t^k + \mathrm{im}\left(\Phi_{\lambda_0}\right) + \sum_{l \in \mathbb{N}^-} \mathrm{im}\left(\Phi_{\mu_l}\right) \cdot t^l$$

by Convention 4.4 and Definition 6.2. Moreover, $\lambda_0 = 0$ by $h = 0$. The rest of the proof proceeds in a fashion analogous to that of Corollary 5.21 (ii). $\square$

7 Completing the induction

In this section, we let $F_0 = C$ and $F_n = C(t_1, \ldots, t_n)$ be a transcendental Liouvillian extension introduced in Definition 1.9. Then $F_i = C(t_1, \ldots, t_i)$ is also a transcendental Liouvillian extension of C for all $i \in [n]$. With Remark 1.6, we state the main result of this paper as follows.

Theorem 7.1. *There exists an algorithm that, for every element $h \in F_n$, constructs a complete reduction $\Phi_{n,h}$ for $(F_n, \mathcal{R}_h)$.*

Proof. We proceed by induction on n. If $n = 0$, then the conclusion holds by Example 3.2.

Let $n > 0$. Assume that, for every $\alpha \in F_{n-1}$, there exists an algorithm that constructs a complete reduction $\Phi_{n-1,\alpha}$ for $(F_{n-1}, \mathcal{R}_\alpha)$, and assume that the map in Hypothesis 4.1 is defined by $\alpha \mapsto \Phi_{n-1,\alpha}$. For $h \in F_n$, let (ξ, η) be the canonical form of h. Then, with the induction hypothesis, one can construct a complete reduction for $(F_n, \mathcal{R}_\xi)$ by either Theorem 5.20 or Theorem 6.15, depending on whether t_n is primitive or hyperexponential over F_{n-1}. Thus, a complete reduction $\Phi_{n,h}$ for $(F_n, \mathcal{R}_h)$ is obtained from Proposition 3.4 (ii). The map in Hypothesis 4.1 is then defined by $h \mapsto \Phi_{n,h}$ for all $h \in F_n$. $\square$

In practice, we need to maintain some initial data in order to avoid choosing elements from a set in different ways (see [21, Remark 2.5]). Let $\Theta_0 = \{1\}$. For each $i \in [n]$, we have an effective basis Θ_i of F_i given in [21, Section 2.2]. We set up a table $\mathbb{T}$ indexed by (i, α), where $i \in [n]$ and $\alpha \in F_i$ is t_i-normalized. The value of (i, α) is an initial (resp. echelon) sequence, where $i \geq 1$ and t_i is primitive (resp. hyperexponential). See Definition A.2 for the notion of initial sequences. The table can be understood as the map in Hypothesis 4.1 with $F = F_i$ for each $i \in [n]$. It is global and will be updated as long as a complete reduction for $(F_i, \mathcal{R}_\alpha)$ is constructed. These considerations lead to an algorithm for computing R-pairs with respect to $\Phi_{n,h}$ given in Theorem 7.1. Below is an outline of the algorithm, in which we initialize the table $\mathbb{T}$ to be empty.

Outline 7.2. *Given $f, h \in F_n$, compute an R-pair of f w.r.t. $\Phi_{n,h}$.*

1. (*Base case*) *If $n = 0$, then an R-pair of f w.r.t. $\Phi_{n,h}$ is $(f/h, 0)$ if $h \neq 0$, and $(0, f)$ if $h = 0$. The algorithm terminates.* (*See Examples 3.2 and 4.2*)

2. (*Preprocessing*)

   (2.1) *Find the canonical form (ξ, η) of h w.r.t. t_n by Algorithm* GKS *in [11, Section 3.2].*

   (2.2) *Compute $(g, r, s) \in F_n \times F_{n-1}\langle t_n \rangle \times S_{t_n,\xi}$ s.t. $\eta f = \mathcal{R}_\xi(g) + r/b + s$ by Algorithm* GKSR *in [11, Section 3.4], where $b = \operatorname{den}(\xi)$ and $S_{t_n,\xi}$ is defined by* (4). (*See (5)*)

   (2.3) *If $r = 0$, then an R-pair of f w.r.t. $\Phi_{n,h}$ is $(\eta^{-1} g, \eta^{-1} s)$. The algorithm terminates.* (*See Propositions 3.6 and 3.4*)

3. (*Recursion*) *Assume that t_n is primitive (resp. hyperexponential) over F_{n-1}.*

   (3.1) *Compute an auxiliary pair (p, q) of r w.r.t. $(F_{n-1}\langle t_n \rangle, \mathcal{P}_\xi)$ by Algorithm A.1 (resp. Algorithm A.7).*

   (3.2) *If $q = 0$, then $(\eta^{-1}(g + p), \eta^{-1} s)$ is a required R-pair. The algorithm terminates.* (*See Propositions 5.5, 6.4 and Remark 3.7*).

   (3.3) (*Updating $\mathbb{T}$*) *If (n, ξ) is an index of $\mathbb{T}$, then set L to be the value of (n, ξ). Otherwise, compute an initial sequence L of I_h by Algorithm A.4 (resp. an echelon sequence E of J_h by Algorithm A.8), and add $(n, \xi) = L$ (resp. $(n, \xi) = E$) to $\mathbb{T}$.*

   (3.4) (*Trivial intersection*) *If $L = \emptyset$ (resp. $E = \emptyset$), then $\left(\eta^{-1}(g + p), \eta^{-1}(q/b + s)\right)$ is a required R-pair of f w.r.t. $\Phi_{n,h}$. The algorithm terminates.* (*See Propositions 5.9, 6.9 and Remark 3.7*)

   (3.5) (*Nontrivial intersection*) *By Algorithm A.6 (resp. Algorithm A.9), compute an element $\tilde{q} \in F_{n-1}\langle t_n \rangle$ and an element $\tilde{r} \in W$ such that $q = \mathcal{P}_\xi(\tilde{q}) + \tilde{r}$, where W is the complement of $\operatorname{im}(\mathcal{P}_\xi)$ in $F_{n-1}\langle t_n \rangle$ induced by L (resp. E) (see Definition A.3 (resp. Definition 6.14)). A required R-pair is $\left(\eta^{-1}(g + p + \tilde{q}), \eta^{-1}(\tilde{r}/b + s)\right)$.* (*See Theorems 5.20, 6.15 and Remark 3.7*)

Remark 7.3. *The complete reduction outlined above depends on algorithms for solving logarithmic derivative recognition problem and parametric logarithmic derivative problem. In practice, such algorithms (see [32, Section 4.3.1]) can be implemented provided that a $\mathbb{Q}$-basis of C is available.*

The two examples below demonstrate the procedure with all steps referenced to Outline 7.2. The first example provides details of Example 1.11.

Example 7.4. *Determine whether* $f = \dfrac{x}{1+\exp(x)} \cdot \exp\left(\dfrac{x}{1+\exp(x)}\right)$ *is a derivative in the transcendental elementary extension* $(\mathbb{C}(x,t,y), d/dx)$*, where* $t = \exp(x)$ *and* $y = \exp\left(\dfrac{x}{1+\exp(x)}\right)$.

Note that $f = gy$*, where* $g = \dfrac{x}{1+\exp(x)}$*. To avoid excessive recursion, we set* $h := y'/y$*, which is equal to* g'*. By the same argument as in Example 3.1, it suffices to compute an R-pair of* g *with respect to the complete reduction* $\Phi_{2,h}$ *for* $(\mathbb{C}(x,t), \mathcal{R}_h)$*.*

Initially, we set $F_0 = \mathbb{C}$*,* $F_1 = F_0(x)$*,* $F_2 = F_1(t)$*, and the table* $\mathbb{T}$ *to be empty.*

Since h *is a derivative in* F_2*, it is* t*-normalized. So* $(\xi, \eta) = (h, 1)$ *in step 2.1. With Convention 4.4 and* $h = \xi$*, we have*

a	b	m	a_m	b_m	a_0	b_0
$1+(1-x)t$	$(1+t)^2$	2	0	1	1	1

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b) + 0$*, where* $r = x + xt$*. Step 2.3 is skipped since* $r \neq 0$*.*

In step 3.1, Algorithm A.7 computes an auxiliary pair $(p, q) = (0, x + xt)$ *of* r*, where* r *belongs to the auxiliary subspace* V_h *associated to* $(F_1[t, t^{-1}], \mathcal{P}_h)$*.*

Step 3.2 is skipped since q *is nonzero. In step 3.3, we find that* $(2, h)$ *is not an index of* $\mathbb{T}$*. Set* $J_h := \operatorname{im}(\mathcal{P}_h) \cap V_h$*. An echelon sequence of* J_h *is constructed as follows.*

First, we obtain that the type of J_h *is* $(0,1), (-1,1)$ *by solving two parametric logarithmic derivative problems. Second, Algorithm A.7 computes the auxiliary pairs of* $\mathcal{P}_h(1)$ *and* $\mathcal{P}_h(t^{-1})$*, which are* $(0, -xt + t + 1)$ *and* $(0, -t - 1 - x)$*, respectively. Third, selecting* xt *as a pivot for* $\mathcal{P}_h(1)$ *and* t *as a pivot for* $\mathcal{P}_h(t^{-1})$*, we obtain the echelon sequence*

$$E:\ (t^{-1}, -t-1-x, t),\ (1, -xt+t+1, xt)$$

of J_h*. Finally, the entry* $(2, h) = E$ *is added to* $\mathbb{T}$*.*

Step 3.4 is skipped since $E \neq \emptyset$*.*

In step 3.5, Algorithm A.9 projects q *to* $\operatorname{im}(\mathcal{P}_h)$ *and the complement of* $\operatorname{im}(\mathcal{P}_h)$ *induced by* E*. The respective projections are* $\mathcal{P}_h(-1 - t^{-1})$ *and* 0*. Then an R-pair of* g *with respect to* $\Phi_{2,h}$ *is* $(-1 - t^{-1}, 0)$*, that is,* $g = \mathcal{R}_h(-1 - t^{-1})$*. It follows from* $h = y'/y$ *and* $f = gy$ *that*

$$f = \left(\left(-1 - t^{-1}\right) y\right)'.$$

In functional notation, we have $\displaystyle\int f = -\big(1 + \exp(-x)\big) \cdot \exp\left(\frac{x}{1+\exp(x)}\right)$.

In the next example, F_n is neither a primitive tower nor a hyperexponential one.

Example 7.5. *Let* $t = \log(x)$*. Determine whether*

$$f = \frac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2+1)} \cdot \exp\left(\int \frac{2x^2 - 2t}{xt^2 + x}\right)$$

is a derivative in $\mathbb{Q}(x, t, f)$*, where the derivation is* d/dx*.*

We rewrite f *in another form so as to make intermediate expressions more compact.*

Let $y = \exp\left(\displaystyle\int \frac{2x^2 - 2t}{xt^2 + x}\right)$*. Then* $f = gy$*, where* $g = \dfrac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2+1)}$*. Moreover,* $\mathbb{Q}(x,t,f) = \mathbb{Q}(x,t,y)$*. By the same reasoning in Example 3.1,* f *belongs to* $\mathbb{Q}(x,t,y)'$ *if and only if there exists an element* $w \in \mathbb{Q}(x,t)$ *such that* $\mathcal{R}_h(w) = g$*, where*

$$h := \frac{y'}{y} = \frac{2x^2 - 2t}{xt^2 + x}.$$

So it suffices to compute an R-pair of g with respect to the complete reduction $\Phi_{2,h}$ for $(\mathbb{Q}(x,t), \mathcal{R}_h)$.

Let $F_0 = \mathbb{Q}$, $F_1 = F_0(x)$ and $F_2 = F_1(t)$. Then h is t-normalized by a straightforward verification. So $(\xi, \eta) = (h, 1)$ in step 2.1. With Convention 4.4 and $\xi = h$, we have

a	b	m	a_m	b_m	a_0	b_0
$-2x^{-1}t + 2x$	t^2+1	2	0	1	$2x$	1

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b)$, where $r = -x^{-2}t^3 - x^{-2}t^2 + (2x+2-x^{-2})t - 2x^3 + x^{-2}$.

In step 3.1, we compute an auxiliary pair

$$(p,q) = \left(x^{-1}t, 2xt - 2x^3\right) \tag{28}$$

of r with respect to $(F_1[t], \mathcal{P}_h)$. During the process of the auxiliary reduction, an R-pair of $\mathrm{lc}_t(r)$ with respect to $\Phi_{1,0}$ is computed. So the entry $(1,0) = 1, (0,1), (0,0), 1$ is added to $\mathbb{T}$.

Note that h is the same as that in Example 5.19, in which the last table presents an initial sequence L with $L[1] = 1, L[2] = (0, x^{-1}), L[3] = (0, 2x^{-1}), L[4] = x^{-1}, L[5] = 2,$

$$L[6] = \left(t^2 - x^2 + 1, 2xt - 2x^3, xt\right), \; L[7] = \left(1, -2x^{-1}t + 2x, x^{-1}t\right),$$

and $L[8] = \left(t, -x^{-1}t^2 + 2xt + x^{-1}, x^{-1}t^2\right)$. Step 3.3 is then completed by adding $(2,h) = L$ to $\mathbb{T}$.

Step 3.4 is skipped because $L \neq \emptyset$.

Let E be the echelon sequence determined by L. In step 3.5, we project q in (28) to $\mathrm{im}(\mathcal{P}_h)$ and W, respectively, where W is the complement of $\mathrm{im}(\mathcal{P}_h)$ induced by E. Since $\deg_t(q) = 1$, no further members in E need to be computed. In fact, q is equal to the second component of $L[6]$. Then the respective projections of q are $\mathcal{P}_h(t^2 - x^2 + 1)$ and 0. It follows that an R-pair of g with respect to $\Phi_{2,h}$ is $(x^{-1}t + t^2 - x^2 + 1, 0)$. In other words, $g = \mathcal{R}_h(x^{-1}t + t^2 - x^2 + 1)$, which implies that $f = (gy)'$. In functional notation, we have

$$\int f = \left(\frac{\log(x)}{x} + \log(x)^2 - x^2 + 1\right) \cdot \exp\left(\int \frac{2x^2 - 2\log(x)}{x\log^2(x) + x}\right).$$

The same integral may also be evaluated using the `int()` command with option `method=parallelrisch` in MAPLE 2026, and the `Integrate[]` command in MATHEMATICA 14.3.

8 Applications

We present two applications of the complete reduction in Theorem 7.1 for the case $h = 0$, in which the Risch operator $\mathcal{R}_0$ is the derivation on a transcendental Liouvillian extension. The first application concerns elementary integration, and the second one addresses reduction-based creative telescoping. To this end, we make the following notational convention throughout this section, because we only consider derivations.

Convention 8.1. *Let $F_0 = C$ and $F_n = C(t_1, \ldots, t_n)$ be a transcendental Liouvillian extension. For all $i \in [n]_0$, Ψ_i stands for the complete reduction $\Phi_{i,0}$ for $(F_i, ')$ given in Theorem 7.1.*

With the above convention, we see that Ψ_0 is the identity map of C and Ψ_n is a complete reduction for $(F_n, ')$. For elementary integration, we need to distinguish primitive generators of F_n from hyperexponential ones.

Set $\mathbb{P} := \{i \in [n] \mid t_i' \in F_{i-1}\}$ and $\mathbb{H} := [n] \setminus \mathbb{P}$. Moreover, define

$$P := \mathrm{span}_C\{\Psi_{i-1}(t_i') \mid i \in \mathbb{P}\} \quad \text{and} \quad H := \mathrm{span}_C\{\Psi_{j-1}(t_j'/t_j) \mid j \in \mathbb{H}\}.$$

Remark 8.2. *If $i \in \mathbb{P}$, the type of* $\operatorname{im}(\mathcal{P}_0) \cap U_0$ *in $F_{i-1}[t_i]$ can be taken as 1 by Example 5.11.*

Some useful properties of remainders are given in the following two technical lemmas.

Lemma 8.3. *For $i \in [n]_0$ and $f \in F_i$, $\Psi_n(f) - \Psi_i(f) \in P$.*

Proof. Let $k \geq i$. By Remark 8.2 and Corollary 5.21, we see that $\Psi_{k+1}(f) - \Psi_k(f) \in P$ if $k+1 \in \mathbb{P}$. By Corollary 6.16, $\Psi_{k+1}(f) - \Psi_k(f) = 0$ if $k+1 \in \mathbb{H}$. Then $\Psi_n(f) - \Psi_i(f) \in P$ by the identity $\Psi_n(f) - \Psi_i(f) = \sum_{k=i}^{n-1} (\Psi_{k+1}(f) - \Psi_k(f))$. ☐

Recall that S_{t_i} stands for the set of t_i-simple elements in $F_{i-1}(t_i)$ for all $i \in [n]$. Set $S := S_{t_1} + \cdots + S_{t_n}$, which is a direct sum. The next definition extends [22, Definition 3.4].

Definition 8.4. *An element of F_n is said to be* simple *if it belongs to S.*

Lemma 8.5. *For every simple element s, we have $s - \Psi_n(s) \in P$.*

Proof. Let $s = \sum_{i \in [n]} s_i$, where $s_i \in S_{t_i}$. By Proposition 3.6 and Remark 3.7, $\Psi_i(s_i) = s_i$ for all $i \in [n]$. Then $s - \Psi_n(s) = \sum_{i \in [n]} (\Psi_i(s_i) - \Psi_n(s_i))$ by $\Psi_n(s) = \sum_{i \in [n]} \Psi_n(s_i)$. The lemma follows from Lemma 8.3. ☐

Next, we define residues of elements in S without reference to any particular generator of F_n.

Definition 8.6. *Let s be a simple element of the form $\sum_{i \in [n]} s_i$ with $s_i \in S_{t_i}$. A* residue of s *is a residue of some s_j with $j \in [n]$ at a normal polynomial of positive degree in $F_{j-1}[t_j]$.*

Residues of simple elements are well-defined by $S = \oplus_{i \in [n]} S_{t_i}$ and Remark 2.4.

We are ready to generalize [21, Theorem 5.3] from primitive towers to transcendental Liouvillian extensions.

Theorem 8.7. *With Convention 8.1, we further assume that C is algebraically closed. Then an element f of F_n has an elementary integral over F_n if and only if*

(i) there exists a simple element s such that $\Psi_n(f) - s \in H + P$, and

(ii) all residues of s belong to C.

Proof. Assume that both (i) and (ii) hold. By (ii) and Lemma 2.6, s has an elementary integral over F_n. Combining with (i), it suffices to show that every element of $H + P$ has an elementary integral over F_n. For $i \in \mathbb{H}$, we have $t_i'/t_i = u_i' + \Psi_{i-1}(t_i'/t_i)$ for some $u_i \in F_{i-1}$, which implies that $\Psi_{i-1}(t_i'/t_i)$ is the derivative of $\log(t_i) - u_i$. For $i \in \mathbb{P}$, we have $t_i' = v_i' + \Psi_{i-1}(t_i')$ for some $v_i \in F_{i-1}$, yielding that $\Psi_{i-1}(t_i') \in F_n'$. Accordingly, $\Psi_n(f)$ has an elementary integral over F_n, and so does f.

Conversely, assume that f has an elementary integral over F_n. Then there exists a C-linear combination g of logarithmic derivatives in F_n such that $f \equiv g \mod F_n'$ by [10, Theorem 5.5.2]. Thus, $\Psi_n(f) = \Psi_n(g)$ by $\Psi_n(F_n') = \{0\}$. It remains to show that (i) and (ii) hold for $\Psi_n(g)$.

By a repeated use of Lemma 2.5 and the logarithmic derivative identity, there exists an element $s \in S$ with merely constant residues, such that $g - s$ is a C-linear combination of t_i'/t_i for $i \in \mathbb{H}$. Then $\Psi_n(g) - \Psi_n(s) \in H + P$ by Lemma 8.3. So $\Psi_n(g) - s \in H + P$ by Lemma 8.5. Both (i) and (ii) hold. ☐

To compute elementary integrals by the above theorem, we set $R_i := t_i \cdot F_{i-1}[t_i]$ if $i \in \mathbb{P}$, and $R_i := (t_i \cdot F_{i-1}[t_i]) + \left(t_i^{-1} \cdot F_{i-1}[t_i^{-1}]\right)$ if $i \in \mathbb{H}$. Moreover, let $R = C + R_1 + \cdots + R_n$. This sum is evidently direct, and so is $R + S$.

Proposition 8.8. *With Convention 8.1 and the notation just introduced, we obtain that*

$$\operatorname{im}(\Psi_n) \subset R \oplus S.$$

Proof. We proceed by induction on n. For $n = 0$, $\operatorname{im}(\Psi_0) = C$ by Example 3.2. So $\operatorname{im}(\Psi_0) \subset R$. The conclusion holds. Assume that $n > 0$ and that the conclusion holds for $n - 1$. We need to consider the cases in which n belongs to either $\mathbb{P}$ or $\mathbb{H}$ separately. Assume that $n \in \mathbb{P}$. Then $\operatorname{im}(\Psi_n) \subset \operatorname{im}(\Psi_{n-1}) + R_n + S_n$ by Corollary 5.21 (ii). Otherwise, $\operatorname{im}(\Psi_n) \subset \operatorname{im}(\Psi_{n-1}) + R_n + S_n$ by Corollary 6.16 (ii). In either case, $\operatorname{im}(\Psi_n) \subset R + S$ holds by the induction hypothesis. □

Next, we outline an algorithm for computing elementary integrals over F_n.

Outline 8.9. *Given $f \in F_n$, determine whether f has an elementary integral over F_n, and computes such an integral if there exists one.*

1. *Compute an R-pair $(g, \Psi_n(f))$. If $\Psi_n(f) = 0$, then $\int f = g$ and the algorithm terminates.*
2. *By Proposition 8.8, we decompose*
   - $\Psi_n(f) = r_f + s_f$,
   - $\Psi_{i-1}(t_i'/t_i) = r_i + s_i$ *if* $i \in \mathbb{H}$, *and* $\Psi_{i-1}(t_i') = r_i + s_i$ *if* $i \in \mathbb{P}$,

   where $r_f, r_i \in R$ and $s_f, s_i \in S$.
3. *Let $z_1, \ldots, z_n$ be constant indeterminates.*

   3.1. *Compute the augmented matrix $(M|v) \in C^{k\times(n+1)}$ of a linear system in $z_1, \ldots, z_n$ induced by the equation $r_f - \sum_{i\in[n]} z_i r_i = 0$.*

   3.2. *Using [21, Algorithm 2.8], compute the augmented matrix $(N|w) \in C^{l\times(n+1)}$ of a linear system in $z_1, \ldots, z_n$ induced by the condition that all residues of $s_f - \sum_{i\in[n]} z_i s_i$ are constants.*

   3.3. *Solve the combined linear system given by $\left(\begin{array}{c|c} M & v \\ N & w \end{array}\right)$. If the system has no solution, then f has no elementary integral over F_n by Theorem 8.7 and the fact that $R + S$ is direct. The algorithm terminates.*
4. *Let $c_1, \ldots, c_n$ be a solution of the combined linear system. Set $s := s_f - \sum_{i\in[n]} c_i s_i$. Then*

$$\begin{aligned}\Psi_n(f) &= r_f + s_f \\ &= \left(r_f - \sum_{i=1}^{n} c_i r_i\right) + \left(s_f - \sum_{i=1}^{n} c_i s_i\right) + \sum_{j\in\mathbb{H}} c_j \Psi_{j-1}\left(\frac{t_j'}{t_j}\right) + \sum_{k\in\mathbb{P}} c_k \Psi_{k-1}\left(t_k'\right) \\ &= s + \sum_{j\in\mathbb{H}} c_j \Psi_{j-1}\left(\frac{t_j'}{t_j}\right) + \sum_{k\in\mathbb{P}} c_k \Psi_{k-1}(t_k').\end{aligned}$$

   The integral of s is elementary over F_n by Lemma 2.6. Let $(q_j, \Psi_{j-1}(t_j'/t_j))$ be an R-pair of t_j'/t_j for all $j \in \mathbb{H}$, and $(q_k, \Psi_{k-1}(t_k'))$ be an R-pair of t_k' for all $k \in \mathbb{P}$. It follows that

$$\int f = g + \int s + \sum_{j\in\mathbb{H}} c_j(\log(t_j) - q_j) + \sum_{k\in\mathbb{P}} c_k(t_k - q_k), \tag{29}$$

   where s is integrated by determining its residues (see [10, Theorem 4.4.3] and [20]).

Note that the integral of s in (29) may involve elements in the algebraic closure of C.

Remark 8.10. *[21, Algorithm 2.8] used in step 3.2 originates from [32, Theorem 3.9]. In practice, such algorithms can be implemented provided that a $\mathbb{Q}$-basis of C is available.*

Example 8.11. *Let us integrate*

$$f := \frac{(\log x + 1)\ \mathrm{Li}(x^c)}{x^{c+1}},$$

where c is a constant indeterminate, $\mathrm{Li}(x)$ is the logarithmic integral, i.e., $\mathrm{Li}(x)' = 1/\log x$.

Let C be the algebraic closure of $\mathbb{Q}(c)$, $t_1 = x$, $t_2 = x^c$, $t_3 = \log(x)$, and $t_4 = \mathrm{Li}(x^c)$. Then $t_1' = 1$, $t_2'/t_2 = c/t_1$, $t_3' = 1/t_1$, $t_4' = t_2/(t_1 t_3)$, and $F_4 := C(t_1, t_2, t_3, t_4)$ is a transcendental Liouvillian extension. Moreover, $f = (t_3 t_4 + t_4)/(t_1 t_2)$.

1. *By the algorithm in Outline 7.2, an R-pair of f with respect to Ψ_4 is $(g, \Psi_4(f))$, where*

$$g = \frac{c t_2 t_3 - c t_3 t_4 - (c+1) t_4}{c^2 t_2} \quad \text{and} \quad \Psi_4(f) = \frac{c+1}{c^2 t_1 t_3}.$$

2. *Projecting remainders $\Psi_4(f)$, $\Psi_0(t_1')$, $\Psi_1(t_2'/t_2)$, $\Psi_2(t_3')$ and $\Psi_3(t_4')$ with respect to $R \oplus S$, we obtain the following table*

	$\Psi_0(t_1')$	$\Psi_1(t_2'/t_2)$	$\Psi_2(t_3')$	$\Psi_3(t_4')$	$\Psi_4(f)$
Proj. in R	$r_1 = 1$	$r_2 = 0$	$r_3 = 0$	$r_4 = 0$	$r_f = 0$
Proj. in S	$s_1 = 0$	$s_2 = c/t_1$	$s_3 = 1/t_1$	$s_4 = t_2/(t_1 t_3)$	$s_f = \Psi_4(f)$

3. *Let z_1, z_2, z_3 and z_4 be constant indeterminates. By making two ansatzes:*
   (i) $r_f = z_1 r_1 + z_2 r_2 + z_3 r_3 + z_4 r_4$, *and*
   (ii) $s_f - z_1 s_1 - z_2 s_2 - z_3 s_3 - z_4 s_4$ *has merely constant residues,*

   we obtain a linear system in z_1, z_2, z_3 and z_4 with augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \end{array}\right).$$

   The system has a solution $(0, 0, 0, 0)$. Hence, f has an elementary integral over F_4.

4. *Computing the residues of s_f yields that*

$$\int f = \frac{1}{c}\log(x) + \frac{c+1}{c^2}\log(\log(x)) - \frac{c \log x + (c+1)}{c^2 x^c}\,\mathrm{Li}(x^c).$$

Example 8.12. *Let $F_0 = \mathbb{C}$ and $F_3 = F_0(t_1, t_2, t_3)$, where*

$$\frac{t_1'}{t_1} = \sqrt{-1}, \quad t_2' = \frac{\sqrt{-1}(t_1^2 + 1)}{t_1^2 - 1} \quad \text{and} \quad \frac{t_3'}{t_3} = \frac{t_2^2 - 1}{2\sqrt{-1}t_2}.$$

Then F_3 is a transcendental Liouvillian extension, where t_1, t_2 and t_3 model

$$\exp(\sqrt{-1}x), \quad \log(\sin x) \quad \text{and} \quad \exp\left(\int \frac{\log(\sin x)^2 - 1}{2\sqrt{-1}\log(\sin x)}\right),$$

respectively. Note that no nonzero constant is a derivative in F_3.

Let us try to integrate

$$f := \frac{(\sqrt{-1}t_2^2 - \sqrt{-1}t_1^2t_2^2 - 2)t_3 + \sqrt{-1}t_1^2t_2 + 3\sqrt{-1}t_2 - 2t_2}{2(t_1^2-1)t_2(t_3+t_2)}$$

over F_3 by the algorithm in Outline 8.9.

1. *An R-pair of f with respect to Ψ_3 is $(0, f)$. So $\Psi_3(f) = f$ and $f \notin F_3'$.*

2. *Projecting the corresponding remainders with respect to $R \oplus S$ yields that*

	$\Psi_0(t_1'/t_1)$	$\Psi_1(t_2')$	$\Psi_2(t_3'/t_3)$	$\Psi_3(f)$
Proj. in R	$r_1 = \sqrt{-1}$	$r_2 = \sqrt{-1}$	$r_3 = t_2/\left(2\sqrt{-1}\right)$	r_f
Proj. in S	$s_1 = 0$	$s_2 = 2\sqrt{-1}/(t_1^2-1)$	$s_3 = -1/\left(2\sqrt{-1}t_2\right)$	s_f

with $r_f = \dfrac{t_2}{2\sqrt{-1}}$ *and* $s_f = \dfrac{1}{(1-t_1^2)t_2} + \dfrac{\sqrt{-1}(t_1^2t_2^2 + t_1^2 - t_2^2 + 3)}{2(t_1^2-1)(t_2+t_3)}.$

3. *Let z_1, z_2, z_3 be three constant indeterminates. Making ansatzes analogous to those in the third step of the above example leads to the linear system in z_1, z_2 and z_3 whose augmented matrix is*

$$\left(\begin{array}{ccc|c} \sqrt{-1} & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \sqrt{-1} & 1 \end{array}\right).$$

Since the system is inconsistent, we conclude that f has no elementary integral over F_3.

Last but not least, we introduce one more application of our complete reduction to creative telescoping. We consider the case in which the integrand contains a shift variable k. More precisely, let $C = \mathbb{C}(k)$, where k is a constant indeterminate. Assume further that the shift operator $\mathcal{S}_k : k \mapsto k+1$ can be extended to F_n, and that the extended operator commutes with the derivation on F_n. For every element $f \in F_n$ and every $i \in \mathbb{N}_0$, one can compute an R-pair (g_i, r_i) of $\mathcal{S}_k^i(f)$ with respect to Ψ_n. Then there exists a recurrence operator $\mathcal{L} \in C[\mathcal{S}_k]^\times$ of order no more than i such that $\mathcal{L}(f) \in F_n'$ if and only if $r_0, r_1, \ldots, r_i$ are C-linearly dependent. This allows us to construct a telescoper for f up to a given order. Unfortunately, we have not found any criterion for the existence of telescopers in such extensions. Discussions related to the construction of telescopers in primitive towers are given in [21, Section 5.2].

We are now ready to supplement the details for Example 1.12.

Example 8.13. *For all $k \in \mathbb{N}$, compute*

$$\int_0^{\frac{\pi}{2}} \cos(2kx)\log(\sin(x))\,dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin(2kx)\log(\sin(x))\,dx.$$

Let $f(k,x) = \exp(2k\sqrt{-1}x)\log(\sin(x))$ and $A(k) = \int_0^{\frac{\pi}{2}} f(k,x)\,dx$. The integrand $f(k,x)$ is not a D-finite function over $\mathbb{C}(k,x)$. The two integrals are the real and imaginary parts of $A(k)$, which are denoted by $R(k)$ and $I(k)$, respectively.

Let $F_0 = \mathbb{C}(k)$ and $F_4 = F_0(t_1, t_2, t_3, t_4)$ with

$$t_1' = 1, \quad \frac{t_2'}{t_2} = \sqrt{-1}, \quad t_3' = \frac{\sqrt{-1}(t_2^2+1)}{t_2^2-1}, \quad \text{and} \quad \frac{t_4'}{t_4} = 2k\sqrt{-1}.$$

The generators t_1 t_2, t_3 and t_4 model x, $\exp(\sqrt{-1}x)$, $\log(\sin(x))$ and $\exp(2k\sqrt{-1}x)$, respectively. Then $f(k,x) = t_3t_4$ and $f(k+1,x) = t_2^2t_3t_4$. Using Ψ_4, we find an R-pair (g_0, r_0) of $f(k,x)$ and an R-pair (g_1, r_1) of $f(k+1,x)$, where

$$(g_0, r_0) = \left(-\frac{\sqrt{-1}(2kt_3 - 1)t_4}{4k^2}, -\frac{t_4}{k(t_2^2 - 1)}\right)$$

and

$$(g_1, r_1) = \left(-\frac{\sqrt{-1}(2k^2t_2^2t_3 + 2kt_2^2t_3 - kt_2^2 - 2k - 2)t_4}{4(k+1)^2k}, -\frac{t_4}{(k+1)(t_2^2 - 1)}\right).$$

Since $(k+1)r_1 - kr_0 = 0$, we have

$$(k+1)f(k+1,x) - kf(k,x) = (k+1)g_1' - kg_0'. \tag{30}$$

Although neither $\lim_{x\to 0^+} g_0$ nor $\lim_{x\to 0^+} g_1$ exists,

$$\lim_{x\to 0^+} ((k+1)g_1 - kg_0) = \frac{\sqrt{-1}(2k+1)}{4k(k+1)}.$$

Integrating both sides of (30) from 0 to $\pi/2$, we have

$$A(k+1) - \frac{k}{k+1}A(k) = \frac{\sqrt{-1}((-1)^k - 2k - 1)}{4k(k+1)^2}. \tag{31}$$

To compute $A(k)$ for all $k \in \mathbb{N}$, we first evaluate $A(1) = \int_0^{\frac{\pi}{2}} \exp(2\sqrt{-1}x)\log(\sin(x))\,dx$. Its integrand is represented by $t_2^2t_3$ in F_3. Using Ψ_3, the integrand has an R-pair

$$\left(-\frac{\sqrt{-1}}{2}t_2^2t_3 + \frac{\sqrt{-1}}{4}t_2^2 + \frac{\sqrt{-1}}{2}t_3 - \frac{t_1}{2}, 0\right).$$

So the integrand belongs to F_3'. It follows that $A(1) = -\pi/4 - \sqrt{-1}/2$.

Taking respective real and imaginary parts of both sides in (31) and $A(1)$, we have

$$\begin{cases} R(k+1) - \dfrac{k}{k+1}R(k) = 0 \\ R(1) = -\dfrac{\pi}{4} \end{cases} \quad \text{and} \quad \begin{cases} I(k+1) - \dfrac{k}{k+1}I(k) = \dfrac{(-1)^k - 2k - 1}{4k(k+1)^2} \\ I(1) = -\dfrac{1}{2}. \end{cases}$$

The recurrence for $R(k)$ together with its initial condition implies $R(k) = -\pi/(4k)$. Using the MATHEMATICA *package* `Sigma` *(see [37]), we get*

$$I(k) = \frac{1 - (-1)^k}{4k^2} - \frac{1}{2k}\left(\sum_{i=1}^{k} \frac{1 - (-1)^i}{i}\right) + \frac{c}{k}$$

for some $c \in \mathbb{C}$. With $I(1) = -1/2$, we find that

$$I(k) = \frac{1 - (-1)^k}{4k^2} - \frac{1}{2k}\left(\sum_{i=1}^{k} \frac{1 - (-1)^i}{i}\right).$$

Remark 8.14. *The result $R(k) = -\pi/(4k)$ is given by Formula 4.384.7 in [27]. Integrals in analogy to $I(k)$ are evaluated in [30] and [32, Chapter 5].*

9 Experiments

We present empirical results about in-field integration obtained from the complete reduction (`CR`) in Theorem 7.1 and the `int()` command in MAPLE without any option. Experiments were carried out with MAPLE 2026 on a computer with iMac CPU 3.6GHz, Intel Core i9, 16GB memory. Our MAPLE scripts for `CR`, the experimental data and timings are contained in a zipped file under the name `ReductionPair.zip` at `https://risc.jku.at/sw/reductionpair/`

Derivations acting on the differential fields in this section are all equal to d/dx. An integrand is the derivative of a rational fraction in a transcendental Liouvillian extension $\mathbb{Q}(t_1, \ldots, t_n)$. A dense (resp. sparse) fraction has numerator and denominator generated by

```
randpoly([t1, t2, ..., tn], degree=d)
```

with option `dense` (resp. the default value, which is sparse).

All integrands in our experiments are derivatives, because `int()` would try to integrate a non-derivative in other closed forms. For each degree d, five derivatives were integrated. Outputs of `CR` were normalized to be rational fractions with coprime numerators and denominators.

Timings were measured by MAPLE CPU time in seconds. Computation would be aborted if one example took more than 3600 seconds. We mark an entry with "$\int$" whenever `int` returned an unevaluated integral. The correctness was verified for each output of `CR`.

In the first suite, we let $n = 3$, $t_1 = x$, $t_2 = \exp(x)$ and $t_3 = \log(\exp(x) + x)$. Fractions were dense in t_1, t_2 and t_3. The average timings are summarized in Table 2.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.01	0.03	0.02	0.16	0.70	2.20	8.01	22.73	61.27	>3600
`int`	0.04	0.08	0.16	0.48	1.44	4.59	27.84	44.41	111.00	>3600

Table 2: Dense fractions in x, $\exp(x)$ and $\log(\exp(x) + x)$

In the second suite, we let $n = 3$, $t_1 = x$, $t_2 = \log(x^2 + 1)$ and $t_3 = \exp(x^2/2)$. Integrands were generated in the same way for the first suite. The average timings are given in Table 3.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.02	0.01	0.05	0.11	0.44	1.61	4.83	12.29	39.43	96.84
`int`	0.04	0.09	0.23	0.74	2.04	6.83	19.78	66.33	176.75	>3600

Table 3: Dense fractions in x, $\log(x^2 + 1)$ and $\exp(x^2/2)$

We let $n = 4$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, $t_3 = \exp(x^2/2)$ and $t_4 = \exp(\exp(x^2/2))$ in the third suite. Rational fractions were sparse, because dense fractions were too huge when $d > 3$. The average timings are given in Table 4.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.01	0.02	0.04	0.07	0.15	0.97	2.42	4.40	30.53	104.30
`int`	0.07	$\int$	0.50	$\int$	1.06	2.84	21.11	14.62	$\int$	219.41

Table 4: Sparse fractions in x, $\log(x^2 + 1)$, $\exp(x^2/2)$ and $\exp(\exp(x^2/2))$

We integrated the derivatives of dense polynomials in t_1, t_2, t_3 and t_4. The average timings are given in Table 5.

d	11	12	13	14	15	16	17	18	19	20
`CR`	0.24	0.31	0.39	0.51	0.62	0.76	0.95	1.17	1.38	1.77
`int`	8.82	13.32	19.64	30.13	53.06	83.45	141.59	257.31	386.16	1242.43

Table 5: Dense polynomials in x, $\log(x^2+1)$, $\exp(x^2/2)$ and $\exp(\exp(x^2/2))$

Lastly, we let $n = 4$, $t_1 = x$, $t_2 = \log(x^2+1)$, $t_3 = \exp(x^2/2)$ and $t_4 = \exp\big(x\log(x^2+1)\big)$. The average timings for integrating the derivatives of sparse fractions and dense polynomials are summarized in Tables 6 and 7, respectively.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.01	0.02	0.05	0.28	0.41	0.19	20.45	0.14	3.43	518.15
`int`	0.08	0.21	0.36	$\int$	$\int$	0.88	41.39	1.44	7.67	1449.17

Table 6: Sparse fractions in x, $\log(x^2+1)$, $\exp(x^2/2)$ and $\exp\big(x\log(x^2+1)\big)$

d	11	12	13	14	15	16	17	18	19	20
`CR`	0.40	0.54	0.67	0.81	1.04	1.27	1.54	1.88	2.22	2.68
`int`	10.06	14.66	21.34	31.84	51.05	71.08	92.80	126.09	174.35	279.45

Table 7: Dense polynomials in x, $\log(x^2+1)$, $\exp(x^2/2)$ and $\exp\big(x\log(x^2+1)\big)$

We did not compare `CR` against `int` for computing elementary integrals, because such a comparison would need an algorithm for determining constant residues. Our MAPLE scripts use an evaluation-based algorithm described in [20], which outperforms resultant-based algorithms developed in 1970's (see [10, Section 4.4]).

10 Concluding remarks

In this paper, we construct a complete reduction for $(F, \mathcal{R}_h)$, where F is a transcendental Liouvillian extension, $h \in F$ and $\mathcal{R}_h$ is the Risch operator associated to h. The complete reduction directly furnishes an algorithm for in-field integration, and leads to a new algorithm for computing elementary integrals over F.

Apart from generalizing the results presented in this paper from transcendental Liouvillian extensions to arbitrary Liouvillian extensions, in which some generators may be algebraic, there remain at least two challenging problems. The first is to develop a reduction-based algorithm for computing telescopers for elements in F without any a priori order bound. Solving this would require a criterion on the existence of telescopers for elements in F. The second problem concerns adapting remainders of the complete reduction for algebraic functions in [16] to enable more efficient computation of elementary integrals over algebraic-function fields.

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A Algorithm descriptions

This appendix is devoted to algorithmic descriptions of some constructive proofs in Sections 5 and 6. Let F be a differential field of characteristic zero, C be the constant subfield of F, and t be a regular monomial over F. For brevity, we use the phrase "the data in Convention 4.4" as shorthand for $F(t)$, h, m, a, b, a_m, b_m, a_0 and b_0 within all algorithm specifications. For $\alpha \in F$, Φ_α stands for the complete reduction for $(F, \mathcal{R}_\alpha)$ in Hypothesis 4.1. Moreover, Θ is an effective basis of F.

A.1 Algorithms in the primitive case

In this subsection, t is primitive over F, $h \in F(t)$ is t-normalized, U_h stands for the auxiliary subspace given in Definition 5.3, and $I_h := \operatorname{im}(\mathcal{P}_h) \cap U_h$.

The first algorithm describes an auxiliary reduction in $F[t]$. It is based on (10) and the proof of Proposition 5.5.

Algorithm A.1. PrimAuxRed

Input: *$f \in F[t]$ and the data in Convention 4.4*

Output: *(g, r), an auxiliary pair of f w.r.t. $(F[t], \mathcal{P}_h)$*

1. $p \leftarrow f$, $(g, r) \leftarrow 0, 0$, $(d, p_d) \leftarrow \deg(p), \operatorname{lc}(p)$
2. while $d \geq m$ do

    if $\nu_\infty(h) < 0$ then $(u, v) \leftarrow a_m^{-1} p_d t^{d-m}$, 0

    else $(\alpha, \beta) \leftarrow$ *an R-pair of p_d w.r.t. Φ_{a_m}*, $(u, v) \leftarrow \alpha t^{d-m}, \beta t^d$

    $(g, r) \leftarrow g + u, r + v$, $p \leftarrow p - \mathcal{P}_h(u) - v$, $(d, p_d) \leftarrow \deg(p), \operatorname{lc}(p)$ (*by (10)*)
3. return $(g, r + p)$

Next, we define a finite sequence L that uniquely determines an echelon sequence of I_h.

Definition A.2. *Let $m = \operatorname{Deg}(h)$ and λ be the type of I_h. We define an* initial sequence L *of I_h as follows.*

1. *If $\lambda = 0$, then set $L := \emptyset$.*

2. Assume that $\lambda \neq 0$. *Fix* $(\tilde{\sigma}, \sigma)$ *and* $(\tilde{\tau}, \tau)$ *to be the first and second R-pairs associated to* $(F[t], \mathcal{P}_h)$*, respectively, and fix an element* $\vartheta_\sigma \in \Theta$ *such that* $\vartheta_\sigma^*(\sigma) \neq 0$.

2.1. If $h \in F$*, then set* L *to be the four-term sequence:*

$$\lambda,\ (\tilde{\sigma}, \sigma),\ (\tilde{\tau}, \tau),\ \vartheta_\sigma. \tag{32}$$

2.2. Assume further that $h \in F(t) \setminus F$.

2.2.1. If $\vartheta_\sigma^*(i\sigma + \tau) \neq 0$ *for all* $i \in \mathbb{N}$*, then set* L *to be the five-term sequence:*

$$\lambda,\ (\tilde{\sigma}, \sigma),\ (\tilde{\tau}, \tau),\ \vartheta_\sigma,\ \left(p_0, \mathcal{P}_h(p_0),\ \vartheta_0 t^{d_0}\right), \tag{33}$$

where the last member is given by Corollary 5.18 (i).

2.2.2. If $\vartheta_\sigma^*(j\sigma + \tau) = 0$ *for some* $j \in \mathbb{N}$ *but* $j\sigma + \tau \neq 0$*, then set* L *to be the seven-term sequence:*

$$\lambda,\ (\tilde{\sigma}, \sigma),\ (\tilde{\tau}, \tau),\ \vartheta_\sigma,\ j,\ \left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right),\ \left(p_j, \mathcal{P}_h(p_j), \vartheta t^{m+j-1}\right), \tag{34}$$

where the last two members are given by Corollary 5.18 (ii).

2.2.3. If $j\sigma + \tau = 0$ *for some* $j \in \mathbb{N}$*, then set* L *to be the* $(j+6)$*-term sequence:*

$$\begin{gathered}\lambda,\ (\tilde{\sigma}, \sigma),\ (\tilde{\tau}, \tau),\ \vartheta_\sigma,\ j,\\ \left(q, \mathcal{P}_h(q), \vartheta t^d\right),\ \left(p_0, \mathcal{P}_h(p_0), \vartheta_0 t^{d_0}\right),\ \left\{\left(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{m+i-1}\right)\right\}_{i \in [j-1]},\end{gathered} \tag{35}$$

where the last $j+1$ *members are given by Corollary 5.18 (iii).*

An initial sequence L with $L \neq \emptyset$ uniquely determines an echelon sequence E of I_h by (13) and Algorithm A.1. Moreover, we do not recompute $\mathcal{P}_h(p_i)$ according to (14).

Definition A.3. *Let* L *and* E *be given above. The complement of* $\operatorname{im}(\mathcal{P}_h)$ *induced by* E *is also called the* complement induced by L.

The next algorithm computes initial sequences.

Algorithm A.4. InitSeq

Input: *the data in Convention 4.4*

Output: *an initial sequence of* I_h

1. $\lambda \leftarrow$ *the type of* I_h, IF $\lambda = 0$ THEN RETURN $\emptyset$ (* $I_h = \{0\}$ *)
2. $(\tilde{\sigma}, \sigma) \leftarrow$ *a first R-pair associated to* $(F[t], \mathcal{P}_h)$
   $(\tilde{\tau}, \tau) \leftarrow$ *a second R-pair associated to* $(F[t], \mathcal{P}_h)$
   $\vartheta_\sigma \leftarrow$ *an element of* Θ *s.t.* $\vartheta_\sigma^*(v) \neq 0$, $\quad L_0 \leftarrow \lambda,\ (\tilde{\sigma}, \sigma),\ (\tilde{\tau}, \tau),\ \vartheta_\sigma$
3. IF $h \in F$ THEN RETURN L_0 (* by Corollary 5.17 and (32) *)
4. $p_0 \leftarrow \lambda$, $\quad (d_0, l_0) \leftarrow \deg(\mathcal{P}_h(p_0)), \operatorname{lc}(\mathcal{P}_h(p_0))$
   $\vartheta_0 \leftarrow$ *an element of* Θ *s.t.* $\vartheta_0^*(l_0) \neq 0$, $\quad \omega_0 \leftarrow \vartheta_0 t^{d_0}$, $\quad j \leftarrow -\vartheta_\sigma^*(\tau)/\vartheta_\sigma^*(\sigma)$, $\quad k \leftarrow -\tau/\sigma$
5. IF $j \notin \mathbb{N}$ THEN RETURN $L_0, (p_0, \mathcal{P}_h(p_0), \omega_0)$ (* by Corollary 5.18 (i) and (33) *)
6. IF $k \notin \mathbb{N}$ THEN *compute* p_j *and* $\mathcal{P}_h(p_j)$ *by* (13), (14) *and Algorithm A.1*
   $l \leftarrow \operatorname{lc}(\mathcal{P}_h(p_j))$, $\quad \vartheta \leftarrow$ *an element of* Θ *s.t.* $\vartheta^*(l) \neq 0$

RETURN $L_0, j, (p_0, \mathcal{P}_h(p_0), \omega_0), (p_j, \mathcal{P}_h(p_j), \vartheta t^{m+j-1})$

*(*by Corollary 5.18 (ii) and (34)*)*

7. 7.1. *compute* $p_1, \mathcal{P}_h(p_1), \ldots, p_{j-1}, \mathcal{P}_h(p_{j-1}), p_j, \mathcal{P}(p_j)$ *by* (13), (14) *and Algorithm A.1*

   7.2. $(q, Q) \leftarrow p_j, \mathcal{P}_h(p_j)$

   7.3. FOR i FROM $j-1$ TO 0 BY -1 DO

   IF $i > 0$ THEN $\omega_i \leftarrow \vartheta_\sigma t^{m+i-1}$

   $c \leftarrow \omega_i^* (\mathcal{P}_h(p_i))^{-1} \cdot \omega_i^*(Q), \quad (q, Q) \leftarrow q - cp_i, Q - c\mathcal{P}_h(p_i)$

   7.4. $(d, l) \leftarrow \deg(Q), \mathrm{lc}(Q)$, $\vartheta \leftarrow$ *an element of* Θ *s.t.* $\vartheta^*(l) \neq 0$

   7.5. RETURN $L_0, j, (q, Q, \vartheta t^d), (p_0, \mathcal{P}_h(p_0), \omega_0), (p_1, \mathcal{P}_h(p_1), \omega_1), \ldots, (p_{j-1}, \mathcal{P}_h(p_{j-1}), \omega_{j-1})$

   *(*by Corollary 5.18 (iii) and (35)*)*

The correctness of this algorithm follows directly from Definition A.2, Corollaries 5.17 and 5.18. Note that j and k in step 4 are equal when both of them are positive integers, justifying the use of j in step 7.

Given an initial sequence L with $L \neq \emptyset$ and a positive integer l, we can compute the first l members of the echelon sequence E by the following algorithm.

Algorithm A.5. PRIMECHSEQ

INPUT: $l \in \mathbb{N}$, *the data in Convention 4.4, and an initial sequence* L *of* I_h *with* $L \neq \emptyset$

OUTPUT: *the first* l *members of the echelon sequence induced by* L

REMARK: *In the following pseudo-code,* $p_0 = L[1]$, p_i *and* $\mathcal{P}_h(p_i)$ *are computed by* (13), (14) *and Algorithm A.1 for* $i > 0$.

1. $\lambda \leftarrow L[1]$, $(\tilde{\sigma}, \sigma) \leftarrow L[2]$, $(\tilde{\tau}, \tau) \leftarrow L[3]$, $\vartheta_\sigma \leftarrow L[4]$
2. IF $\mathrm{len}(L) = 4$ THEN RETURN $\{(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{i-1})\}_{i \in [l]}$ (*by *Corollary 5.17* and (32)*)
3. IF $\mathrm{len}(L) = 5$ THEN RETURN $L[5]$, $\{(p_i, \mathcal{P}_h(p_i), \vartheta_\sigma t^{m+i-1})\}_{i \in [l-1]}$

   (**by Corollary 5.18 (i) and* (33)*)
4. $j \leftarrow L[5]$
5. IF $\mathrm{len}(L) = 7$ AND $j\sigma + \tau \neq 0$ THEN $(q_1, Q_1, \omega_1) \leftarrow L[6]$

   FOR i FROM 2 TO l DO

   IF $i \neq j+1$ THEN $(q_i, Q_i, \omega_i) \leftarrow p_{i-1}, \mathcal{P}_h(p_{i-1}), \vartheta_\sigma t^{m+i-2}$ ELSE $(q_i, Q_i, \omega_i) \leftarrow L[7]$

   RETURN $(q_1, Q_1, \omega_1), (q_2, Q_2, \omega_2), \ldots, (q_l, Q_l, \omega_l)$ (**by Corollary 5.18 (ii) and* (34)*)
6. 6.1. FOR i FROM 1 TO $j+1$ DO $(q_i, Q_i, \omega_i) \leftarrow L[5+i]$

   6.2. FOR i FROM $j+2$ TO l DO $(q_i, Q_i, \omega_i) \leftarrow p_{i-1}, \mathcal{P}_h(p_{i-1}), \vartheta_\sigma t^{m+i-2}$

   6.3. RETURN $(q_1, Q_1, \omega_1), (q_2, Q_2, \omega_2), \ldots, (q_l, Q_l, \omega_l)$ (**by Corollary 5.18 (iii) and* (35)*)

The following algorithm computes the respective projections of a given element in $F[t]$ to $\mathrm{im}(\mathcal{P}_h)$ and its complement induced by an initial sequence.

Algorithm A.6. PRIMPROJ

INPUT: $f \in F[t]$, *the data in Convention 4.4, and an initial sequence* L *of* I_h *with* $L \neq \emptyset$

OUTPUT: $(g, r) \in F[t] \times W$ *s.t.* $f = \mathcal{P}_h(g) + r$, *where* W *is the complement induced by* L

1. $\lambda \leftarrow L[1]$ *(*retrieve the type*)*

2. $(g, r) \leftarrow$ *an auxiliary pair of f computed by Algorithm A.1* (*auxiliary reduction*)
3. IF $\lambda = 0$ OR $r = 0$ THEN RETURN (g, r) (* $I_h = \{0\}$ *or the trivial projection**)
4. *use Algorithm A.5 to compute the first k members $(q_1, Q_1, \omega_1), \ldots, (q_k, Q_k, \omega_k)$ in the echelon sequence E induced by L s.t. all remaining members in E are of degrees* $> \deg(r)$
5. FOR i FROM k TO 1 BY -1 DO $c \leftarrow \omega_i^*(Q_i)^{-1}\omega_i^*(r)$, $(g, r) \leftarrow g + cq_i, r - cQ_i$ (*elimination*)
6. RETURN (g, r)

The first four steps of the above algorithm are evidently correct. Step 5 is justified by Lemma 2.13 and its proof. In step 6, r belongs to the complement of $\mathrm{im}(\mathcal{P}_h)$ induced by L, because $\omega_i^*(r) = 0$ for all $i \in [k]$ and the pivot of the lth member does not appear in r for all $l > k$, as guaranteed by our choice of k in step 4.

A.2 Algorithms in the hyperexponential case

In this subsection, t is hyperexponential over F, $h \in F(t)$ is t-normalized, V_h is the auxiliary space given in Definition 6.2, and $J_h = \mathrm{im}(\mathcal{P}_h) \cap V_h$. The elements λ_k and μ_l of F are defined in Remark 6.1 (ii) and (iv), respectively.

The first algorithm implements the auxiliary reduction in $F[t, t^{-1}]$. It is based on the congruences in (26) and (27), and the proof of Proposition 6.4.

Algorithm A.7. HYPEREXPAUXRED
INPUT: $f \in F[t, t^{-1}]$ *and the data in Convention 4.4*
OUTPUT: (g, r), *an auxiliary pair of f w.r.t.* $(F[t, t^{-1}], \mathcal{P}_h)$

1. $p \leftarrow f^+$, $(k, p_k) \leftarrow \mathrm{hdeg}(p), \mathrm{hc}(p)$, $q \leftarrow f^-$, $(l, q_l) \leftarrow \mathrm{tdeg}(q), \mathrm{tc}(q)$, $(g, r) \leftarrow 0, 0$
2. WHILE $k \geq m$ DO

   IF $\nu_\infty(h) < 0$ THEN $(u, v) \leftarrow a_m^{-1} p_k t^{k-m}, 0$, $g \leftarrow g + u$

   ELSE

   $(\alpha, \beta) \leftarrow$ *an R-pair of p_k w.r.t.* $\Phi_{\lambda_{k-m}}$, $(u, v) \leftarrow \alpha t^{k-m}, \beta t^k$, $(g, r) \leftarrow g + u$, $r + v$

   $p \leftarrow p - \mathcal{P}_h(u) - v$, $(k, p_k) \leftarrow \mathrm{hdeg}(p), \mathrm{hc}(p)$ (*by (26)*)
3. WHILE $l < 0$ DO

   IF $\nu_t(h) < 0$ THEN $(u, v) \leftarrow a_0^{-1} q_l t^l, 0$, $g \leftarrow g + u$

   ELSE

   $(\alpha, \beta) \leftarrow$ *an R-pair of $b_0^{-1} q_l$ w.r.t.* Φ_{μ_l}, $(u, v) \leftarrow \alpha t^l, b_0 \beta t^l$, $(g, r) \leftarrow g + u, r + v$

   $q \leftarrow q - \mathcal{P}_h(u) - v$, $(l, q_l) \leftarrow \mathrm{tdeg}(q), \mathrm{tc}(q)$ (*by (27) *)
4. RETURN $(g, r + p + q)$

Next, we determine whether $J_h = \{0\}$, and find an echelon sequence of J_h with respect to $(F[t, t^{-1}], \mathcal{P}_h)$ if $J_h \neq \{0\}$. This algorithm is based on Propositions 6.12 and 6.13.

Algorithm A.8. HYPEREXPECHSEQ
INPUT: *the data in Convention 4.4*
OUTPUT: $\emptyset$ *if* $J_h = \{0\}$, *and an echelon sequence of* J_h, *otherwise*

1. $M \leftarrow$ *the type of* J_h
2. IF $M = 0$ THEN RETURN $\emptyset$ (* $J_h = \{0\}$ *)

3. (*$\dim(J_h) = 1$*)

IF $\operatorname{len}(M) = 1$ THEN $(k, \sigma) \leftarrow M[1]$ (*find the type*)

$(g, r) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(\sigma t^k)$ *computed by Algorithm A.7*

$(p, P) \leftarrow ut^k - g, r$, $\vartheta \leftarrow$ *an element of* Θ *s.t.* $\operatorname{hc}(P) \notin \ker(\vartheta^*)$

RETURN $\left(p, P, \vartheta t^{\operatorname{hdeg}(P)}\right)$ (*by Proposition 6.13 (i)*)

4. (*$\dim(J_h) = 2$*) $(k, \sigma) \leftarrow M[1]$, $(l, \tau) \leftarrow M[2]$ (*find the type*)

4.1. $(g_k, r_k) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(\sigma t^k)$ *computed by Algorithm A.7*

$(g_l, r_l) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(\tau t^l)$ *computed by Algorithm A.7*

$(p_k, P_k) \leftarrow \sigma t^k - g_k, r_k$, $(p_l, P_l) \leftarrow \tau t^l - g_l, r_l$ (*by Proposition 6.12 (ii)*)

4.2. $d_k \leftarrow \operatorname{hdeg}(P_k)$, $\vartheta_k \leftarrow$ *an element of* Θ *s.t.* $\operatorname{hc}(P_k) \notin \ker(\vartheta_k^*)$, $\omega_k \leftarrow \vartheta_k t^{d_k}$

$c \leftarrow \omega_k^*(P_k)^{-1} \omega_k^*(P_l)$, $(q, Q) \leftarrow p_l - cp_k, P_l - cP_k$, $d \leftarrow \operatorname{hdeg}(Q)$

$\vartheta \leftarrow$ *an element of* Θ *s.t.* $\operatorname{hc}(Q) \notin \ker(\vartheta^*)$, $\omega \leftarrow \vartheta t^d$, RETURN $(q, Q, \omega), (p_k, P_k, \omega_k)$

(*by Proposition 6.13 (ii)*)

The last algorithm computes the respective projections of an element in $F[t, t^{-1}]$ to $\operatorname{im}(\mathcal{P}_h)$ and the complement induced by an echelon sequence of J_h. It is simpler than Algorithm A.6, because an echelon sequence of J_h has at most two members. We therefore present only its formal specification.

Algorithm A.9. HYPEREXPPROJ

INPUT: $f \in F[t, t^{-1}]$, *the data in Convention 4.4, and an echelon sequence* E *of* J_h

OUTPUT: $(g, r) \in F[t, t^{-1}] \times W$ *s.t.* $f = \mathcal{P}_h(g) + r$, *where* W *is the complement induced by* E