A Unified Reduction for Hypergeometric and q-Hypergeometric Creative Telescoping*

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Dedicated to Professors George Andrews and Bruce Berndt for their 85th birthdays

Abstract

We adapt the theory of normal and special polynomials from symbolic integration to the summation setting, and then built up a general framework embracing both the usual shift case and the q-shift case. In the context of this general framework, we develop a unified reduction algorithm, and subsequently a creative telescoping algorithm, applicable to both hypergeometric terms and their q-analogues. Our algorithms allow to split up the usual shift case and the q-shift case only when it is really necessary, and thus instantly reveal the intrinsic differences between these two cases. Computational experiments are also provided.

1 Introduction

Hypergeometric summation and its q-analogue appear frequently in combinatorics [8]. These are sums whose summands are (q-)hypergeometric terms, typically involving rational functions, geometric terms, factorial terms, binomial coefficients, and so on. Given a (q-)hypergeometric sum, an important problem is to decide whether the sum admits a "closed form". A prominent technique for tackling such a problem is the method of *creative telescoping*, also known as *Zeilberger's algorithm* in the hypergeometric case and *q-Zeilberger's algorithm* in the q-hypergeometric case. This method was first pioneered by Zeilberger [46, 47, 48] in the 1990s and has now become the primary technique for definite summation and integration.

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In the case of summation, the method of creative telescoping takes a summand $f(x, y)$ and looks for polynomials $c_0, c_1, \ldots, c_\rho$ in x only, not all zero, and another term $g(x, y)$ in the same class as $f(x, y)$ such that

$$c_0(x)f + c_1(x)S_x(f) + \cdots + c_\rho(x)S_x^\rho(f) = S_y(g) - g, \tag{1}$$

where S_x and S_y denote (q-)shift operators with respect to x and y, respectively. The number ρ may or may not be part of the input. If such $c_0, c_1, \ldots, c_\rho$ and g exist, then the nonzero recurrence operator $L = c_0 + c_1S_x + \cdots + c_\rho S_x^\rho$ is called a *telescoper* for f and the term g is called the *certificate* for L. From the relation (1), one can derive a recurrence equation admitting the given definite sum as a solution. With this equation at hand, we are then able to evaluate the sum by some other available algorithms (for example, cf. [42, 2]) or (dis-)prove an already given identity by substitution and initial-value checking. As an example, let us consider the q-Chu-Vandermonde identity in the form

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} = \begin{bmatrix} b+n \\ n \end{bmatrix}_q,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise} \end{cases}$$

is the Gaussian binomial coefficient and $(q;q)_k = \prod_{i=1}^{k}(1-q^i)$ is the q-Pochhammer symbol with $(q;q)_0 = 1$. Let $f_{n,k}$ denote the summand on the left-hand side of the identity, and let $x = q^n$ and $y = q^k$. The method of creative telescoping then constructs a telescoper $L = q(1 - q^{b+1}x) - q(1 - qx)S_x$ for $f_{n,k}$ and a corresponding certificate

$$g_{n,k} = \frac{q^2x(y-1)^2}{-qx+y} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2},$$

where $S_x(f_{n,k}) = f_{n+1,k}$. Thus (1) becomes

$$q(1 - q^{b+1}x)f_{n,k} - q(1 - qx)f_{n+1,k} = g_{n,k+1} - g_{n,k}.$$

Summing over k from zero to n on both sides, along with a subsequent simplification, delivers the recurrence equation

$$q(1 - q^{b+1}x)F_n - q(1 - qx)F_{n+1} = 0 \quad \text{with } F_n = \sum_{k=0}^{n} f_{n,k}. \tag{2}$$

Using a q-analogue of Pascal's formula (cf. [28, Exercises 1.2 (v)]), we see that the right-hand side of the identity $\begin{bmatrix} b+n \\ n \end{bmatrix}_q$ satisfies the same recurrence equation. The correctness of the q-Chu-Vandermonde identity is finally confirmed by checking the equality of initial values at $n = 0$. Similar reasoning processes apply to most of the summation identities listed in [28, Appendix II].

Over the past 35 years, a variety of generalizations and improvements of creative telescoping have been developed. As outlined in the introduction of [23], we can distinguish four generations among them. The first generation was based on elimination techniques. The second generation starts with Zeilberger's algorithm, and uses the idea of parametrizing an algorithm for indefinite summation (or integration). The third generation was initiated by Apagodu and Zeilberger, and mainly applies a second-generation algorithm by hand to a generic input so as to reduce the problem to solving linear systems. Due to the efficiency and practicability, the algorithms of the second generation have been implemented in many computer algebra systems, including Maple and Mathematica, and are widely used in proving identities from combinatorics, the theory of partitions or physics, etc. More details can be found in [43] for the first two generations and in [39, 9] for the third one.

In terms of the fourth generation, reduction methods currently provide the state of the art for constructing telescopers (see [17] and the references therein). These convert a given summand $f(x, y)$ into some sort of reduced form $\operatorname{red}(f)$ modulo all terms that are y-differences of other terms. The object is then to continually take shifts $f, S_x(f), S_x^2(f), \ldots$ and, by reduction methods, convert these to reduced forms $\operatorname{red}(f), \operatorname{red}(S_x(f)), \operatorname{red}(S_x^2(f)), \ldots$ until we find a set of polynomials $c_0(x), c_1(x), \ldots, c_\rho(x)$, not all zero, such that

$$c_0(x) \operatorname{red}(f) + c_1(x) \operatorname{red}(S_x(f)) + \cdots + c_\rho(x) \operatorname{red}(S_x^\rho(f)) = 0.$$

This will give rise to a telescoper $L = c_0 + c_1 S_x + \cdots + c_\rho S_x^\rho$ for f. The termination of the above process is guaranteed by known existence criteria for telescopers (for example, see [1] for the hypergeometric case and [26] its q-analogue).

Compared with previous generations, the key advantage of the fourth generation is that it separates the computation of the c_i from the computation of the g in (1), and thus enables one to find a telescoper without also necessarily computing a corresponding certificate. This is desirable in a typical situation where only the telescoper is of interest and its size is much smaller than the size of the certificate. For instance, in the above example of the q-Chu-Vandermonde identity, due to the natural boundary of Gaussian binomial coefficients, we could have directly used the telescoper to obtain the recurrence equation (2) without knowing the certificate. So far this approach has been worked out for various classes of functions, including rational functions [10, 13, 22], hyperexponential functions [11], algebraic functions [24], D-finite functions [21, 12, 33, 19], hypergeometric terms [23, 35], P-recursive sequences [32, 14, 20], and so on. One goal of the present paper is to further enlarge the list and include another important class of q-hypergeometric terms.

q-Hypergeometric terms are just slight adaptations of the usual ones by essentially promoting involved variables to exponents of an additional parameter q. One of the reasons for interest in q-analogues is that, due to the extra parameter q, they have many counting interpretations which are useful in combinatorics and analysis; see the classic books [6, 7] for many interesting applications in combinatorics, analysis and elsewhere in mathematics. Very often, techniques for handling the usual case carry over to the q-analogue with some subtle modifications (cf. [37, 40]). Rather than working out these modifications individually, we aim to set up a general framework which combines both the usual shift case and the q-shift case so as to reveal more profound reasons for this phenomenon. The foundation of this framework is the theory of normal and special polynomials adapted from symbolic integration [15]. With this theory, every rational function can be uniquely decomposed as a sum of the polynomial, normal, and special parts. As indicated by Lemma 2.7, a major difference from the q-shift case to the usual one lies in the appearance of nontrivial special polynomials (and thus possibly nontrivial special parts). This results in the Laurent polynomial reduction, instead of the usual polynomial reduction, for q-hypergeometric terms in [27]. In order to eliminate this discrepancy, we introduce the notion of standard rational functions (see Definition 3.7), which enables us to transform a nontrivial special part to a “simpler” term that can be tackled simultaneously with the polynomial part. In this way, we unify the reduction processes for hypergeometric terms and their q-analogues, and subsequently obtain a unified creative telescoping algorithm. This algorithm extends the one developed in [23] for the usual hypergeometric terms by including the q-shift case, and it shares the important feature of the reduction-based approach that the computation of a telescoper is separated from that of its certificate.

The remainder of the paper proceeds as follows. Some basic notions and results are recalled in the next section. In particular, we briefly translate the theory of normal and special polynomials from symbolic integration to the summation setting. Using this theory, we present in Section 3 a reduction algorithm which brings every (q-)hypergeometric term to a reduced form.

These reduced forms are shown in Section 4 to be well-behaved with respect to taking linear combinations. Based on the reduction algorithm developed previously, Section 5 describes an algorithm for constructing telescopers for (q-)hypergeometric terms, followed in Section 6 by an experimental comparison between our algorithms and the built-in algorithms of Maple.

2 Preliminaries

Throughout the paper, let $\mathbb{F}$ be a field of characteristic zero, and $\mathbb{F}(y)$ be the field of rational functions in y over $\mathbb{F}$. Let σ_y be an $\mathbb{F}$-automorphism of $\mathbb{F}(y)$. The pair $(\mathbb{F}(y), \sigma_y)$ is called a *difference field*. By [45, Theorem 6.2.3], there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}) \quad \text{with } a, b, c, d \in \mathbb{F} \text{ and } ad - bc \neq 0,$$

such that $\sigma_y = \varphi_A$, where φ_A denotes the $\mathbb{F}$-automorphism of $\mathbb{F}(y)$ defined by

$$\varphi_A(f(y)) = f\left(\frac{ay+b}{cy+d}\right) \quad \text{for all } f \in \mathbb{F}(y).$$

Let

$$A_u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \text{ with } q \in \mathbb{F} \setminus \{0\}.$$

We call σ_y the *usual shift operator* if $\sigma_y = \varphi_{A_u}$, and call it the *q-shift operator* if $\sigma_y = \varphi_{A_q}$.

According to the discussions in [25, §2, Page 323], we see that the difference field $(\mathbb{F}(y), \sigma_y)$ is actually isomorphic to either the difference field $(\mathbb{F}(y), \varphi_{A_u})$ or the difference field $(\mathbb{F}(y), \varphi_{A_q})$. In other words, there exists an automorphism ϕ of $\mathbb{F}(y)$ such that

$$\text{either} \quad \phi \circ \sigma_y = \varphi_{A_u} \circ \phi \quad \text{or} \quad \phi \circ \sigma_y = \varphi_{A_q} \circ \phi. \tag{3}$$

Now let R be a ring extension of $\mathbb{F}(y)$ and assume that the relation (3) remains in R, where $\sigma_y, \varphi_{A_u}, \varphi_{A_q}$ are extended to be monomorphisms of R, and ϕ is extended to be an automorphism of R. An element $c \in R$ is called a *constant* if $\sigma_y(c) = c$. All constants in R form a subring of R, denoted by C_R.

Definition 2.1. *An invertible element T of R is called a* σ_y-hypergeometric term *over $\mathbb{F}(y)$ if $\sigma_y(T) = rT$ for some $r \in \mathbb{F}(y)$. We call r the* σ_y-quotient *of T.*

Clearly, every nonzero rational function in $\mathbb{F}(y)$ is σ_y-hypergeometric. A σ_y-hypergeometric term is also called a *hypergeometric term* if σ_y is the usual shift operator, and a *q-hypergeometric term* if σ_y is the q-shift operator. Typical examples are given by a Pochhammer symbol $(a)_y = a(a+1)\cdots(a+y-1)$ $(y > 0)$ being a hypergeometric term in y and its q-analogue $(q;q)_k = \prod_{i=1}^{k}(1-q^i)$ being a q-hypergeometric term in q^k.

Two σ_y-hypergeometric terms are called *similar* over $\mathbb{F}(y)$ if one can be obtained from the other by multiplying a rational function in $\mathbb{F}(y)$. A σ_y-hypergeometric term T is said to be *σ_y-summable* if there exists another σ_y-hypergeometric term G such that $T = \Delta_y(G)$, where Δ_y denotes the difference of σ_y and the identity map of R. It is readily seen that two σ_y-hypergeometric terms T and G satisfying $T = \Delta_y(G)$ are similar.

Let σ_1, σ_2 be any two monomorphisms of R with $\phi \circ \sigma_1 = \sigma_2 \circ \phi$ for some automorphism ϕ of R whose restriction to $\mathbb{F}(y)$ is an automorphism of $\mathbb{F}(y)$. Then a term T in R is σ_1-hypergeometric if and only if $\phi(T)$ is σ_2-hypergeometric. Moreover, the problem of determining whether a σ_1-hypergeometric term T is σ_1-summable is equivalent to that of determining whether the σ_2-hypergeometric term $\phi(T)$ is σ_2-summable. We thus conclude from (3) that determining the σ_y-summability of a σ_y-hypergeometric term amounts to determining the usual summability of a hypergeometric term or the q-summability of a q-hypergeometric term.

For simplicity, in the rest of the paper, we assume throughout that, when restricted to $\mathbb{F}(y)$, the $\mathbb{F}$-automorphism σ_y is either the usual shift operator such that $\sigma_y(y) = y + 1$ or the q-shift operator such that $\sigma_y(y) = qy$, where $q \in \mathbb{F}$ is neither zero nor a root of unity. These two cases will be later referred to as the usual shift case and the q-shift case, respectively. We remark that in the case when σ_y is the q-shift operator and q is further assumed to be a root of unity, the σ_y-summability problem is closely related to the additive version of Hilbert's Theorem 90 (see [38, Theorem 6.3, Page 290]), and will be left for future research.

2.1 The canonical representation

Let T be a σ_y-hypergeometric term. A key idea on determining the σ_y-summability of a given σ_y-hypergeometric term T is to write it into a multiplicative decomposition $T = fH$, where $f \in \mathbb{F}(y)$ and H is a σ_y-hypergeometric term enjoying some nice properties (cf. [3, 5]). Then determining whether T is σ_y-summable amounts to finding a rational function $g \in \mathbb{F}(y)$ such that

$$fH = \Delta_y(gH), \text{ or equivalently, } f = K\sigma_y(g) - g \text{ with } K = \frac{\sigma_y(H)}{H}. \tag{4}$$

For a nonzero rational function K in $\mathbb{F}(y)$, we follow [27] to define a linear map $\Delta_K = K\sigma_y - \mathbf{1}$ from $\mathbb{F}(y)$ to itself which maps $r \in \mathbb{F}(y)$ to $K\sigma_y(r) - r$. Note that the image of Δ_K, denoted by $\operatorname{im}(\Delta_K)$, is an $\mathbb{F}$-linear subspace of $\mathbb{F}(y)$. It then follows from (4) that fH is σ_y-summable if and only if $f \in \operatorname{im}(\Delta_K)$. In this way, the main object has been reduced from σ_y-hypergeometric terms to the well-studied class of rational functions.

In the following, we adapt the notion of normal and special polynomials from symbolic integration [15] into our setting, so as to obtain a canonical representation of a rational function.

Definition 2.2. *A polynomial $p \in \mathbb{F}[y]$ is said to be σ_y-*normal *if $\gcd(p, \sigma_y^\ell(p)) = 1$ for any nonzero integer ℓ, and σ_y-*special *if $p \mid \sigma_y^\ell(p)$ for some nonzero integer ℓ.*

Note that σ_y-normal polynomials are also called σ_y-free polynomials in the literature. Recall that two polynomials a, b in $\mathbb{F}[y]$ are *associates* if $a = c\,b$ for $c \in \mathbb{F}$. Since σ_y preserves the degree of the input polynomial, a polynomial $p \in \mathbb{F}[y]$ is σ_y-special if and only if p is an associate of $\sigma_y^\ell(p)$ for some nonzero integer ℓ, which happens if and only if p is an associate of $\sigma_y^\ell(p)$ for some positive integer ℓ. It is readily seen that σ_y-normal (resp. σ_y-special) polynomials remain to be σ_y-normal (resp. σ_y-special) under the application of the automorphism σ_y. A polynomial is not necessarily σ_y-normal or σ_y-special, but an irreducible polynomial $p \in \mathbb{F}[y]$ must be either σ_y-normal or σ_y-special, since $\gcd(p, \sigma_y^\ell(p))$ for any integer ℓ is a factor of p. Evidently, all elements in $\mathbb{F}$ are σ_y-special, and $p \in \mathbb{F}[y]$ is both σ_y-normal and σ_y-special if and only if $p \in \mathbb{F} \setminus \{0\}$.

Definition 2.3. *Two polynomials $a, b \in \mathbb{F}[y]$ are said to be σ_y-*coprime *if $\gcd(a, \sigma_y^\ell(b)) = 1$ for any nonzero integer ℓ.*

Note that in the above definition, we did not require that two σ_y-coprime polynomials be coprime. In analogy to [15, Theorem 3.4.1], we describe the multiplicative properties of σ_y-special and σ_y-normal polynomials.

Proposition 2.4.

(i) Any finite product of σ_y-normal and pairwise σ_y-coprime polynomials in $\mathbb{F}[y]$ is σ_y-normal. Any factor of a σ_y-normal polynomial in $\mathbb{F}[y]$ is σ_y-normal.

(ii) Any finite product of σ_y-special polynomials in $\mathbb{F}[y]$ is σ_y-special. Any factor of a nonzero σ_y-special polynomial in $\mathbb{F}[y]$ is σ_y-special.

Proof. (i) Let $p_1, \dots, p_m \in \mathbb{F}[y]$ be σ_y-normal and such that $\gcd(p_i, \sigma_y^k(p_j)) = 1$ for all $i, j, k \in \mathbb{Z}$ with $1 \le i < j \le m$ and $k \neq 0$, and let $p = \prod_{i=1}^m p_i$. Then for any nonzero integer ℓ, we have

$$\gcd(p, \sigma_y^\ell(p)) = \gcd(p_1 \cdots p_m, \sigma_y^\ell(p_1) \cdots \sigma_y^\ell(p_m)) \;\Big|\; \prod_{i=1}^m \gcd(p_i, \sigma_y^\ell(p_i)) = 1,$$

where the last equality follows by the σ_y-normality of each p_i. Thus $\gcd(p, \sigma_y^\ell(p)) = 1$, that is, p is σ_y-normal.

Let $p \in \mathbb{F}[y]$ be σ_y-normal and write $p = ab$ where $a, b \in \mathbb{F}[y]$. Since p is σ_y-normal, we have

$$\gcd(p, \sigma_y^\ell(p)) = \gcd(ab, \sigma_y^\ell(a)\sigma_y^\ell(b)) = 1 \quad \text{for all } \ell \in \mathbb{Z} \setminus \{0\}.$$

Thus $\gcd(a, \sigma_y^\ell(a)) = 1$ for all $\ell \in \mathbb{Z} \setminus \{0\}$, which implies that a is σ_y-normal.

(ii) Let $a, b \in \mathbb{F}[y]$ be two σ_y-special polynomials. Then there exist positive integers ℓ_1, ℓ_2 and elements $c_1, c_2 \in \mathbb{F}$ such that $\sigma_y^{\ell_1}(a) = c_1 a$ and $\sigma_y^{\ell_2}(b) = c_2 b$. Thus

$$\sigma_y^{\ell_1\ell_2}(ab) = \sigma_y^{\ell_1\ell_2}(a)\sigma_y^{\ell_1\ell_2}(b) = (c_1 a)(c_2 b) = c_1 c_2 ab.$$

So $ab \mid \sigma_y^{\ell_1\ell_2}(ab)$, that is, ab is σ_y-special. The first assertion of part (ii) then follows by induction.

Let p be a nonzero σ_y-special polynomial in $\mathbb{F}[y]$. There is nothing to show if $p \in \mathbb{F}$. Assume that $p \notin \mathbb{F}$ and let $a \in \mathbb{F}[y]$ be an irreducible factor of p. Then there exists a nonzero integer ℓ such that p and $\sigma_y^\ell(p)$ are associates, and thus $a \mid \sigma_y^\ell(p)$. It follows that there exists an irreducible factor $a_1 \in \mathbb{F}[y]$ of p such that a is an associate of $\sigma_y^\ell(a_1)$. Applying the same argument to a_1, we get an irreducible factor $a_2 \in \mathbb{F}[y]$ of p such that a_1 is an associate of $\sigma_y^\ell(a_2)$. Thus, a is an associate of $\sigma_y^{2\ell}(a_2)$. Continuing in this pattern, we obtain a sequence of irreducible factors $\{a_1, a_2, \dots\} \subseteq \mathbb{F}[y]$ of p such that

$$a \text{ is an associate of } \sigma_y^{k\ell}(a_k) \quad \text{for all } k = 1, 2, \dots$$

Since p has only finitely many irreducible factors, there exist two integers i, j with $j > i \ge 1$ such that $a_i = a_j$. Since a is an associate of both $\sigma_y^{i\ell}(a_i)$ and $\sigma_y^{j\ell}(a_j)$, we see that $a \mid \sigma_y^{(j-i)\ell}(a)$. Notice that $j - i > 0$. So $(j - i)\ell \neq 0$. By definition, a is σ_y-special. Since a is arbitrary, we know that every irreducible factor of p is σ_y-special. Let now $b \in \mathbb{F}[y]$ be any factor of p. If $b \in \mathbb{F}$, then b is σ_y-special by definition. Otherwise, b is a nonempty finite product of irreducible factors of p, so it is σ_y-special by the first assertion of part (ii). □

We can separate the σ_y-normal and σ_y-special components of a polynomial in $\mathbb{F}[y]$.

Definition 2.5. *Let $p \in \mathbb{F}[y]$. We say that $p = p_s p_n$ is a σ_y-*splitting factorization *of p if $p_s, p_n \in \mathbb{F}[y]$, p_s is σ_y-special, and every irreducible factor of p_n is σ_y-normal.*

A consequence of Proposition 2.4 is that we always have $\gcd(p_s, p_n) = 1$ in a σ_y-splitting factorization $p = p_s p_n$ of $p \in \mathbb{F}[y]$, and such a factorization is unique up to multiplication by units in $\mathbb{F}$. Clearly, a full irreducible factorization of p yields a σ_y-splitting factorization of p. It is not obvious to us (but it would be interesting to see) whether such a σ_y-splitting factorization can be computed by the gcd computation only, like in the differential case (cf. [15, §3.5]).

For a nonzero polynomial $p \in \mathbb{F}[y]$, its degree in y (or y-degree) is denoted by $\deg_y(p)$. We will follow the convention that $\deg_y(0) = -\infty$. We assume throughout that the numerator and denominator of a rational function in $\mathbb{F}(y)$ are always coprime. A rational function in $\mathbb{F}(y)$ is said to be *proper* if the y-degree of its numerator is less than that of its denominator.

We can now define a canonical representation of rational functions in $\mathbb{F}(y)$. Let f be a rational function in $\mathbb{F}(y)$ with denominator d and let $d = d_s d_n$ be a σ_y-splitting factorization of d. Then there are unique $p, a, b \in \mathbb{F}[y]$ such that $\deg_y(a) < \deg_y(d_s)$, $\deg_y(b) < \deg_y(d_n)$, and

$$f = p + \frac{a}{d_s} + \frac{b}{d_n}.$$

We call this decomposition, which is unique, the *σ_y-canonical representation* of f, and the components $p, a/d_s, b/d_n$ the *polynomial part*, the *special part*, the *normal part* of f, respectively. Note that the special and normal parts of a rational function in $\mathbb{F}(y)$ are always proper.

In the usual shift and the q-shift cases, we are able to characterize all possible σ_y-special irreducible polynomials. To this end, we need the following simple lemma.

Lemma 2.6. *Let f be a rational function in $\mathbb{F}(y)$.*

(i) If $f(y+\ell) = f(y)$ for some nonzero integer ℓ, then $f \in \mathbb{F}$.

(ii) If $f(q^\ell y) = cf(y)$ for some nonzero integer ℓ and $c \in \mathbb{F}$, then $f/y^k \in \mathbb{F}$ for some $k \in \mathbb{Z}$. In particular, $c = q^{\ell k}$ if $f \neq 0$.

Proof. (i) This is exactly [4, Lemma 2].

(ii) It is trivial when $f = 0$. Assume that f is nonzero and $f(q^\ell y) = cf(y)$ for some $\ell \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{F}$. Write $f = a/d$ with $a, d \in \mathbb{F}[y] \setminus \{0\}$ and $\gcd(a, d) = 1$. It then follows from $f(q^\ell y) = cf(y)$ that

$$a(q^\ell y) = c_1 a(y) \quad \text{and} \quad d(q^\ell y) = c_2 d(y),$$

where $c_1, c_2 \in \mathbb{F}$ with $c_1/c_2 = c$. Let $a = \sum_{i=0}^{k_1} a_i y^i$ with $k_1 \in \mathbb{N}$, $a_i \in \mathbb{F}$ and $a_{k_1} \neq 0$. By comparing the coefficients of both sides of $a(q^\ell y) = c_1 a(y)$, we conclude that

$$a = a_{k_1} y^{k_1} \quad \text{and} \quad c_1 = q^{\ell k_1}.$$

Similarly, we have $d = d_{k_2} y^{k_2}$ and $c_2 = q^{\ell k_2}$ for some $k_2 \in \mathbb{N}$ and $d_{k_2} \in \mathbb{F} \setminus \{0\}$. Letting $k = k_1 - k_2 \in \mathbb{Z}$, we obtain that $f/y^k \in \mathbb{F}$ and $c = q^{\ell k}$. $\square$

The following lemma gives the desired characterization.

Lemma 2.7. *Let p be a polynomial in $\mathbb{F}[y]$.*

(i) In the usual shift case, p is σ_y-special if and only if $p \in \mathbb{F}$.

(ii) In the q-shift case, p is σ_y-special if and only if p is an associate of y^k for some $k \in \mathbb{N}$.

Proof. (i) The sufficiency is evident by definition. For the necessity, assume that p is σ_y-special. Then there exist $\ell \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{F}$ such that $\sigma_y^\ell(p) = c\,p$. Since σ_y is the usual shift operator, $\sigma_y^\ell(p)$ and p have the same leading coefficient with respect to y. Thus $c = 1$ and $\sigma_y^\ell(p) = p$. It follows from part (i) of Lemma 2.6 that $p \in \mathbb{F}$.

(ii) Assume that $p = cy^k$ for some $c \in \mathbb{F}$ and $k \in \mathbb{N}$. Since σ_y is the q-shift operator, we have $\sigma_y(p) = c\,q^k y^k$. So $p \mid \sigma_y(p)$, and thus p is σ_y-special by definition. Conversely, assume that p is σ_y-special. Then there exist $\ell \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{F}$ such that $\sigma_y^\ell(p) = c\,p$. The assertion follows from part (ii) of Lemma 2.6. $\square$

2.2 Kernels, shells and σ_y-factorizations

Recall that a polynomial p in $\mathbb{F}[y]$ is said to be *monic* if its leading coefficient with respect to y is one, and *q-monic* if $p(0) = 1$ (cf. [40]). Unifying the usual shift and the q-shift cases, we say that a polynomial p in $\mathbb{F}[y]$ is *σ_y-monic* if it is monic in the former case or q-monic in the latter one. A rational function in $\mathbb{F}(y)$ is *σ_y-monic* if both its numerator and denominator are σ_y-monic. By a factor of a rational function in $\mathbb{F}(y)$, we mean a factor of either its numerator or its denominator. Let $f \in \mathbb{F}(y)$ be σ_y-monic. Lemma 2.7 then tells us that all irreducible factors of f are σ_y-normal. Moreover, $\sigma_y^\ell(f)$ for all $\ell \in \mathbb{Z}$ is again σ_y-monic.

Based on [5, 26], a nonzero rational function in $\mathbb{F}(y)$ with numerator u and denominator v is said to be *σ_y-reduced* if u and v are σ_y-coprime. For a nonzero rational function $f \in \mathbb{F}(y)$, there exist two nonzero rational functions K, S in $\mathbb{F}(y)$ with K being σ_y-reduced such that

$$f = K\frac{\sigma_y(S)}{S}.$$

Such a pair (K, S) will be called a *rational normal form* (or an *RNF* for short) of f. Moreover, we call K a *kernel* and S a corresponding *shell* of f. These quantities can be constructed by gcd-calculations (cf. [5, 26]).

It is known that a rational function in $\mathbb{F}(y)$ is a Laurent polynomial in y if its denominator is a power of y. All Laurent polynomials in $\mathbb{F}(y)$ form a subring, which is denoted by $\mathbb{F}[y, y^{-1}]$. Let f be a nonzero Laurent polynomial in $\mathbb{F}[y, y^{-1}]$. Then it can be written in the form $f = \sum_{i=m}^{n} c_i y^i$, where $m, n \in \mathbb{Z}$ with $m \leq n$ and $c_m, c_{m+1}, \ldots, c_n \in \mathbb{F}$ with $c_m c_n \neq 0$. We call n the *head degree* of f and m the *tail degree* of f, which are denoted by $\operatorname{hdeg}(f)$ and $\operatorname{tdeg}(f)$, respectively. By convention, we have $\operatorname{hdeg}(0) = -\infty$ and $\operatorname{tdeg}(0) = +\infty$.

We will also consider the ring of Laurent polynomials in σ_y over $\mathbb{Z}$, denoted by $\mathbb{Z}[\sigma_y, \sigma_y^{-1}]$. Let p be a nonzero polynomial in $\mathbb{F}[y]$ and let $\alpha = \sum_{i=m}^{n} k_i \sigma_y^i \in \mathbb{Z}[\sigma_y, \sigma_y^{-1}]$. We define

$$p^{\alpha} = \prod_{i=m}^{n} \sigma_y^i(p)^{k_i}.$$

Clearly, p^{α} is a polynomial if and only if α belongs to $\mathbb{N}[\sigma_y, \sigma_y^{-1}]$.

According to [36, Definition 11] and [5, Definition 1], two polynomials $a, b \in \mathbb{F}[y]$ are *σ_y-equivalent* if a is an associate of $\sigma_y^{\ell}(b)$ for some integer ℓ. Evidently, this gives an equivalence relation, and the σ_y-equivalence of two polynomials can be easily recognized by comparing coefficients. Let f be a rational function in $\mathbb{F}(y)$. By computing σ_y-splitting factorizations of its numerator and denominator, and grouping together all σ_y-normal irreducible factors that are σ_y-equivalent, we can decompose f as

$$f = f_s\, p_1^{\alpha_1} \ldots p_m^{\alpha_m}, \tag{5}$$

where $f_s \in \mathbb{F}(y)$ whose numerator and denominator are both σ_y-special, $m \in \mathbb{N}$, each $\alpha_i \in \mathbb{Z}[\sigma_y, \sigma_y^{-1}] \backslash \{0\}$, each $p_i \in \mathbb{F}[y]$ is σ_y-monic and irreducible, and the p_i are pairwise σ_y-inequivalent. We call (5) a *σ_y-factoriztion* of f. Note that such a factorization is not unique since there are many possibilities to express each component $p_i^{\alpha_i}$. Nevertheless, for each fixed p_i, the corresponding exponent α_i in (5) is unique as p_i is σ_y-normal and $\mathbb{F}[y]$ is a unique factorization domain, and we will then call α_i the *σ_y-exponent* of p_i in p.

The following describes a useful property of σ_y-reduced rational functions, which is equivalent to [23, Lemma 2.2] in the usual shift case and [27, Proposition 3.2] in the q-shift case.

Proposition 2.8. *Let r be a σ_y-reduced rational function in $\mathbb{F}(y)$, and assume that $r = \sigma_y(f)/f$ for some $f \in \mathbb{F}(y) \setminus \{0\}$. Then both the numerator and denominator of f are σ_y-special. Moreover, r is equal to one in the usual shift case, or it is a power of q in the q-shift case.*

Proof. Assume that f admits a σ_y-factorization of the form (5) and suppose that $m > 0$. Since $r = \sigma_y(f)/f$, we get

$$r = \frac{\sigma_y(f_s)}{f_s} \prod_{i=1}^{m} p_i^{\beta_i},$$

where $\beta_i = \sigma_y \alpha_i - \alpha_i \neq 0$ for all $i = 1, \ldots, m$. Notice that the total sum of all coefficients of each β_i with respect to σ_y is zero. So each β_i has both positive and negative coefficients, which contradicts with the assumption that r is σ_y-reduced. Therefore, $m = 0$ and then $f = f_s$, implying that both the numerator and denominator of f are σ_y-special. The second assertion immediately follows by Lemma 2.7. □

The above property enables us to derive the following equivalence characterization of *rational σ_y-hypergeometric terms*, that is, terms of the form cf, where $c \in C_R \setminus \{0\}$ and $f \in \mathbb{F}(y)$.

Corollary 2.9. *Let $T \in R$ be a σ_y-hypergeometric term whose σ_y-quotient has a kernel K.*

(i) In the usual shift case, T is rational if and only if $K = 1$.

(ii) In the q-shift case, T is rational if and only if K is a power of q.

Proof. Let S be a shell of $\sigma_y(T)/T$ so that $\sigma_y(T)/T = K\sigma_y(S)/S$. Assume that $T = cf$ for some $c \in C_R \setminus \{0\}$ and $f \in \mathbb{F}(y)$. Then $\sigma_y(T)/T = \sigma_y(f)/f = K\sigma_y(S)/S$. Thus $K = \sigma_y(r)/r$, where $r = f/S \in \mathbb{F}(y)$. Since K is σ_y-reduced, we see from Proposition 2.8 that K is equal to one in the usual shift case or it is a power of q in the q-shift case. The necessities of both parts (i) and (ii) thus follow.

Conversely, assume that $K = 1$ in the usual shift case or $K = q^k$ for some $k \in \mathbb{Z}$ in the q-shift case. By taking $r = 1$ in the former case or $r = y^k$ in the latter one, we obtain that $K = \sigma_y(r)/r$ in either case. Notice that $\sigma_y(T)/T = K\sigma_y(S)/S$. Thus $T/(rS)$ is a nonzero constant, say c, of the ring R. It follows that $T = c\,rS$. □

As mentioned in the paragraph right after Corollary 3.2 in [27], the ring R can be chosen using Picard-Vessiot extensions (cf. [16, 31]) so that C_R coincides with the field $\mathbb{F}$ if $\mathbb{F}$ is further assumed to be algebraically closed.

3 A unified reduction

Let T be a σ_y-hypergeometric term whose σ_y-quotient has an RNF (K, S). According to [5, 26], T admits a *multiplicative decomposition* SH, where H is another σ_y-hypergeometric term with σ_y-quotient K. We are going to reduce the shell S modulo $\operatorname{im}(\Delta_K)$ to a rational function $r \in \mathbb{F}(y)$, which is minimal in some sense. This reduction gives rise to an additive decomposition $T = \Delta_y(gH) + rH$ for some $g \in \mathbb{F}(y)$. The minimality of r will then establish a σ_y-summability criterion which says that T is σ_y-summable if and only if $r = 0$ (see Theorem 3.18).

Using an arbitrary RNF, the task described above can be accomplished by a shell reduction enhanced with a polynomial reduction in both the usual shift case [23] and the q-shift case [27]. The difference is that, in the q-shift case, we need to reduce Laurent polynomials instead of polynomials, which complicates the steps for the polynomial reduction. In order to force the q-shift case to be in line with the usual shift case as much as possible, inspired by [41], we will introduce the notion of σ_y-standard rational functions (see Definition 3.7) and further show that every nonzero rational function in $\mathbb{F}(y)$ has a σ_y-standard kernel (see Proposition 3.16). For a σ_y-hypergeometric term whose σ_y-quotient has a σ_y-standard kernel K and a corresponding shell S, the main steps of our reduction algorithm are as follows: first write the shell S in its canonical representation and then perform successive reductions each of which brings the individual part to a "simple" one, until the remaining rational function is "minimal".

3.1 Normal reduction

In this subsection, we aim at reducing the normal part of a rational function to a "simple" one. The main strategy differs slightly from the method presented in [27, §4] in that it employs the so-called strong σ_y-factorization, rather than an arbitrary one, so as to make the process more concise and more trackable. The following definition is useful in justifying the simplicity.

Definition 3.1. *Let $K \in \mathbb{F}(y)$ with numerator u and denominator v. A nonzero polynomial $p \in \mathbb{F}[y]$ is said to be* strongly coprime *with K if $\gcd(u, \sigma_y^{\ell}(p)) = \gcd(v, \sigma_y^{-\ell}(p)) = 1$ for all $\ell \in \mathbb{N}$.*

The next lemma is used to verify the minimality of our reduction algorithm, which applies to both the usual shift and the q-shift cases, and thus extends [27, Lemma 4.1].

Lemma 3.2. *Let $K \in \mathbb{F}(y)$ with denominator v, and let $h \in \mathbb{F}(y)$ be a rational function whose denominator d is σ_y-normal and strongly coprime with K. Assume that there are $\tilde{h} \in \mathbb{F}(y)$ and $p \in \mathbb{F}[y]$ such that*

$$h - \tilde{h} + \frac{p}{v} \in \operatorname{im}(\Delta_K). \tag{6}$$

Then the degree of d is no more than that of the denominator of $\tilde{h}$.

Proof. Write $K = u/v$, where $u \in \mathbb{F}[y]$ with $\gcd(u, v) = 1$. By (6), there exists $g \in \mathbb{F}(y)$ such that

$$h - \tilde{h} + \frac{p}{v} = K\sigma_y(g) - g.$$

Multiplying both sides by v yields

$$v(h - \tilde{h}) - (u\sigma_y(g) - vg) = -p \in \mathbb{F}[y]. \tag{7}$$

There is nothing to show if $d \in \mathbb{F}$. Assume that $d \notin \mathbb{F}$ and let $a \in \mathbb{F}[y]$ be an irreducible factor of d with multiplicity k. Since d is σ_y-normal, a is σ_y-normal as well by Proposition 2.4 (i). Notice that all irreducible factors of d are mutually σ_y-inequivalent. So it suffices to show that there exists an integer ℓ such that $\sigma_y^\ell(a)^k$ divides the denominator $\tilde{d}$ of $\tilde{h}$. Suppose that a^k does not divide $\tilde{d}$, otherwise we have done. Notice that $a \nmid v$ as d is strongly coprime with K. It thus follows from (7) that a^k divides either e or $\sigma_y(e)$, where e is the denominator of g.

If $a^k \mid e$, then there exists an integer $\ell \geq 1$ such that $\sigma_y^{\ell-1}(a)^k \mid e$ but $\sigma_y^\ell(a)^k \nmid e$ since a is σ_y-normal. Thus we have $\sigma_y^\ell(a)^k \mid \sigma_y(e)$. Since d is σ_y-normal and strongly coprime with K, neither d nor u is divisible by $\sigma_y^\ell(a)$. Therefore, we conclude from (7) that $\sigma_y^\ell(a)^k$ divides $\tilde{d}$.

If $a^k \mid \sigma_y(e)$, then there exists an integer $\ell \leq -1$ such that $\sigma_y^\ell(a)^k \mid e$ but $\sigma_y^{\ell-1}(a)^k \nmid e$ since a is σ_y-normal. Thus we have $\sigma_y^\ell(a)^k \nmid \sigma_y(e)$. Similarly, neither d nor v is divisible by $\sigma_y^\ell(a)$, because d is σ_y-normal and strongly coprime with K. Therefore, $\sigma_y^\ell(a)^k$ divides $\tilde{d}$ by (7). □

Let $p \in \mathbb{F}[y]$ be σ_y-normal and irreducible, and $\alpha \in \mathbb{N}[\sigma_y, \sigma_y^{-1}] \setminus \{0\}$. Let $K \in \mathbb{F}(y)$ be σ_y-reduced with numerator u and denominator v. Let $\lambda, \mu \in \mathbb{N}[\sigma_y, \sigma_y^{-1}]$ be the σ_y-exponents of p in u and v, respectively. Since K is σ_y-reduced, at least one of λ and μ is zero, that is, we have $\lambda\mu = 0$. Define

$$\ell = \begin{cases} \operatorname{tdeg}(\mu) - 1 & \text{if } \lambda = 0, \\ \operatorname{hdeg}(\lambda) + 1 & \text{otherwise.} \end{cases} \tag{8}$$

Then we rewrite

$$p^\alpha = (p^{\sigma_y^\ell})^{\sigma_y^{-\ell}\alpha},$$

where $p^{\sigma_y^\ell}$ is strongly coprime with K and $\sigma_y^{-\ell}\alpha \in \mathbb{N}[\sigma_y, \sigma_y^{-1}] \setminus \{0\}$. Note that $p^{\sigma_y^\ell}$ is again σ_y-normal and irreducible. This implies that every σ_y-normal and irreducible polynomial in $\mathbb{F}[y]$ can be transformed to one which is σ_y-equivalent to the original polynomial and is strongly coprime with K. Such a transformation enables us to obtain a more structured σ_y-factorization of a given polynomial in $\mathbb{F}[y]$.

Consider now a polynomial $p \in \mathbb{F}[y]$ with a σ_y-factorization $p = p_s \prod_{i=1}^m p_i^{\alpha_i}$ and let $K \in \mathbb{F}(y)$ be a σ_y-reduced rational function. For each factor p_i, we transform it to one which is strongly coprime with K using the procedure described in the preceding paragraph. By relabeling all the resulting factors, we finally arrive at the following decomposition (with a slight abuse of notation)

$$p = p_s \, p_1^{\alpha_1} \dots p_m^{\alpha_m}, \tag{9}$$

where $m \in \mathbb{N}$ and

- $p_s \in \mathbb{F}[y]$ is σ_y-special;

- each $p_i \in \mathbb{F}[y]$ is σ_y-monic, irreducible and strongly coprime with K;
- the p_i are pairwise σ_y-inequivalent;
- each α_i is in $\mathbb{N}[\sigma_y, \sigma_y^{-1}] \setbackslash \{0\}$.

We will call (9) a *strong σ_y-factorization* of p with respect to K.

Before turning to the general case, we first perform the normal reduction "locally".

Lemma 3.3. *Let $K \in \mathbb{F}(y)$ with denominator v, and let $f \in \mathbb{F}(y)$ be a nonzero proper rational function with denominator $d^{k\sigma_y^\ell}$, where $k \in \mathbb{Z}^+$, $\ell \in \mathbb{Z}$ and $d \in \mathbb{F}[y]$ is strongly coprime with K. Then there exist $g \in \mathbb{F}(y)$ and $a, b \in \mathbb{F}[y]$ with $\deg_y(a) < k\deg_y(d)$ such that*

$$f = \Delta_K(g) + \frac{a}{d^k} + \frac{b}{v}. \tag{10}$$

Proof. Write $K = u/v$ and $f = c/d^{k\sigma_y^\ell}$, where $u, c \in \mathbb{F}[y]$ with $\gcd(u, v) = 1$ and $\gcd(c, d^{k\sigma_y^\ell}) = 1$. If $\ell = 0$ then letting $g = 0$, $a = c$ and $b = 0$ immediately yields the assertion. Now assume that ℓ is nonzero. So it is either positive or negative.

If $\ell > 0$, then $\gcd(u, d^{k\sigma_y^\ell}) = 1$ since d is strongly coprime with K. Using the extended Euclidean algorithm, we can find $s, t \in \mathbb{F}[y]$ with $\deg_y(s) < k\deg_y(d)$ such that

$$vc = su + td^{k\sigma_y^\ell}.$$

Multiplying both sides by $1/(vd^{k\sigma_y^\ell})$ gives

$$f = \frac{vc}{vd^{k\sigma_y^\ell}} = K\frac{s}{d^{k\sigma_y^\ell}} + \frac{t}{v}.$$

Adding and subtracting $\sigma_y^{-1}(s)/d^{k\sigma_y^{\ell-1}}$ to the right-hand side, we get

$$f = K\sigma_y\Big(\frac{\sigma_y^{-1}(s)}{d^{k\sigma_y^{\ell-1}}}\Big) - \frac{\sigma_y^{-1}(s)}{d^{k\sigma_y^{\ell-1}}} + \frac{\sigma_y^{-1}(s)}{d^{k\sigma_y^{\ell-1}}} + \frac{t}{v} = \Delta_K(g_0) + \tilde{f} + \frac{t}{v}, \tag{11}$$

where $g_0 = \sigma_y^{-1}(s)/d^{k\sigma_y^{\ell-1}}$ and $\tilde{f} = \sigma_y^{-1}(s)/d^{k\sigma_y^{\ell-1}}$. If $\tilde{f} = 0$, the proof is then concluded by letting $g = g_0 = 0$, $a = 0$ and $b = t$. Otherwise, $\tilde{f}$ is a nonzero proper rational function with denominator $d^{k\sigma_y^{\ell-1}}$, so by induction on ℓ, we can find $\tilde{g} \in \mathbb{F}(y)$ and $a, \tilde{b} \in \mathbb{F}[y]$ with $\deg_y(a) < k\deg_y(d)$ such that

$$\tilde{f} = \Delta_K(\tilde{g}) + \frac{a}{d^k} + \frac{\tilde{b}}{v},$$

which, along with (11), establishes (10) with $g = g_0 + \tilde{g}$ and $b = t + \tilde{b}$.

If $\ell < 0$, then $\gcd(v, d^{k\sigma_y^{\ell+1}}) = 1$ since d is strongly coprime with K. Again, we can employ the extended Euclidean algorithm to find $s, t \in \mathbb{F}[y]$ with $\deg_y(s) < k\deg_y(d)$ such that

$$u\sigma_y(c) = sv + td^{k\sigma_y^{\ell+1}}.$$

Multiplying both sides by $1/(vd^{k\sigma_y^{\ell+1}})$ gives

$$K\sigma_y(f) = \frac{u\sigma_y(c)}{vd^{k\sigma_y^{\ell+1}}} = \frac{s}{d^{k\sigma_y^{\ell+1}}} + \frac{t}{v}.$$

Adding and subtracting $K\sigma_y(f)$ to f, we get

$$f = K\sigma_y(-f) - (-f) + K\sigma_y(f) = K\sigma_y(-f) - (-f) + \frac{s}{d^{k\sigma_y^{\ell+1}}} + \frac{t}{v} = \Delta_K(g_0) + \tilde{f} + \frac{t}{v}, \tag{12}$$

where $g_0 = -f$ and $\tilde{f} = s/d^{k\sigma_y^{\ell+1}}$. If $\tilde{f} = 0$, then the assertion follows by letting $g = g_0 = -f$, $a = 0$ and $b = t$. Otherwise, $\tilde{f}$ is a nonzero proper rational function with denominator $d^{k\sigma_y^{\ell+1}}$, so by induction on ℓ, we can find $\tilde{g} \in \mathbb{F}(y)$ and $a, \tilde{b} \in \mathbb{F}[y]$ with $\deg_y(a) < k \deg_y(d)$ such that

$$\tilde{f} = \Delta_K(\tilde{g}) + \frac{a}{d^k} + \frac{\tilde{b}}{v},$$

which, along with (12), establishes (10) with $g = g_0 + \tilde{g}$ and $b = t + \tilde{b}$. □

Lemma 3.4. *Let $K \in \mathbb{F}(y)$ with denominator v, and let $f \in \mathbb{F}(y)$ be a nonzero proper rational function with denominator d^α, where $d \in \mathbb{F}[y]$ is σ_y-normal and strongly coprime with K and $\alpha \in \mathbb{N}[\sigma_y, \sigma_y^{-1}] \setminus \{0\}$. Then there exist $g \in \mathbb{F}(y)$, $a, b \in \mathbb{F}[y]$ and $k \in \mathbb{N}$ with $\deg_y(a) < k \deg_y(d)$ such that*

$$f = \Delta_K(g) + \frac{a}{d^k} + \frac{b}{v}. \tag{13}$$

Proof. Assume that $\alpha = \sum_{i=m}^{n} k_i \sigma_y^i$, where $m, n \in \mathbb{Z}$, $m \leq n$, $k_i \in \mathbb{N}$ and $k_m k_n \neq 0$. Since d is σ_y-normal, the polynomials $d^{\sigma_y^m}, d^{\sigma_y^{m+1}}, \ldots, d^{\sigma_y^n}$ are pairwise coprime. Then f admits a partial fraction decomposition $f = \sum_{i=m}^{n} f_i$, where each f_i is either zero or a nonzero proper rational function in $\mathbb{F}(y)$ with denominator $d^{k_i \sigma_y^i}$. Applying Lemma 3.3 to each nonzero f_i yields

$$f_i = \Delta_K(g_i) + \frac{a_i}{d^{k_i}} + \frac{b_i}{v},$$

where $g_i \in \mathbb{F}(y)$, $a_i, b_i \in \mathbb{F}[y]$ and $\deg_y(a_i) < k_i \deg_y(d)$. Summing all these equations up, we thus obtain (13) for some $g \in \mathbb{F}(y)$, $a, b \in \mathbb{F}[y]$ and $k \in \mathbb{N}$ satisfying $\deg_y(a) < k \deg_y(d)$. □

The main result of this subsection is given below.

Theorem 3.5. *Let $K \in \mathbb{F}(y)$ be σ_y-reduced with denominator v, and let $f \in \mathbb{F}(y)$ be a proper rational function whose denominator does not have any nontrivial σ_y-special factors. Then there exist $g, h \in \mathbb{F}(y)$ and $b \in \mathbb{F}[y]$ such that*

$$f = \Delta_K(g) + h + \frac{b}{v}, \tag{14}$$

and h is proper whose denominator is σ_y-normal and strongly coprime with K. Moreover, the denominator of h has minimal y-degree in the sense that if there exists another triple $(\tilde{g}, \tilde{h}, \tilde{b})$ with $\tilde{g}, \tilde{h} \in \mathbb{F}(y)$ and $\tilde{b} \in \mathbb{F}[y]$ such that

$$f = \Delta_K(\tilde{g}) + \tilde{h} + \frac{\tilde{b}}{v}, \tag{15}$$

then the y-degree of the denominator of h is no more than that of $\tilde{h}$. In particular, h is equal to zero if $f \in \operatorname{im}(\Delta_K)$.

Proof. If $f = 0$, then the assertion is evident by letting $g = h = b = 0$. Assume that f is nonzero and write $f = a/d$ with $a, d \in \mathbb{F}[y]$, $\gcd(a, d) = 1$ and $\deg_y(a) < \deg_y(d)$. Let $d = d_s \prod_{i=1}^{m} d_i^{\alpha_i}$ be a strong σ_y-factorization of d with respect to K, where $d_s \in \mathbb{F}$ by assumption. Then f has a partial fraction decomposition

$$f = \sum_{i=1}^{m} f_i, \tag{16}$$

where $f_i \in \mathbb{F}(y)$ is a nonzero proper rational function with denominator $d_i^{\alpha_i}$ for $i = 1, \ldots, m$. For all $i = 1, \ldots, m$, we can apply Lemma 3.4 to f_i to find $g_i \in \mathbb{F}(y)$, $a_i, b_i \in \mathbb{F}[y]$ and $k_i \in \mathbb{N}$ with $\deg_y(a_i) < k_i \deg_y(d_i)$ such that

$$f_i = \Delta_K(g_i) + \frac{a_i}{d_i^{k_i}} + \frac{b_i}{v}. \tag{17}$$

Then (14) follows by letting $g = \sum_{i=1}^m g_i$, $h = \sum_{i=1}^m a_i/d_i^{k_i}$ and $b = \sum_{i=1}^m b_i$. Note that the irreducible polynomials $d_1, \ldots, d_m$ are σ_y-normal and mutually σ_y-inequivalent. So they are two by two σ_y-coprime, and thus the denominator of h is σ_y-normal by Proposition 2.4 (i). Because $d_1, \ldots, d_m$ are all strongly coprime with K, so is the denominator of h. Moreover, h is proper since all the $a_i/d_i^{k_i}$ are proper.

It remains to show that the y-degree of the denominator of h is minimal. Assume that there exist $\tilde{g}, \tilde{h} \in \mathbb{F}(y)$ and $\tilde{b} \in \mathbb{F}[y]$ such that (15) holds. Then by (14), we have

$$h - \tilde{h} + \frac{b - \tilde{b}}{v} \in \operatorname{im}(\Delta_K).$$

It follows from Lemma 3.2 that the y-degree of the denominator of h is no more than that of $\tilde{h}$. Now assume that $f \in \operatorname{im}(\Delta_K)$. Then (15) holds with $\tilde{h} = \tilde{b} = 0$ and thus $h \in \mathbb{F}[y]$ by the minimality. Since h is proper, it must be zero. □

The proof of Theorem 3.5 induces an algorithm as follows.

NormalReduction. Given a σ_y-reduced rational function $K \in \mathbb{F}(y)$ with denominator v, and a proper rational function $f \in \mathbb{F}(y)$ whose denominator does not have any nontrivial σ_y-special factors, compute two rational functions $g, h \in \mathbb{F}(y)$ and a polynomial $b \in \mathbb{F}[y]$ such that (14) holds and h is proper whose denominator is σ_y-normal and strongly coprime with K.

1. If $f = 0$, then set $g = 0, h = 0, b = 0$, and return.
2. Compute a strong σ_y-factorization $d = d_s \prod_{i=1}^m d_i^{\alpha_i}$ of the denominator d of f with respect to K.
3. Compute the partial fraction decomposition (16) of f with respect to $d = d_s \prod_{i=1}^m d_i^{\alpha_i}$.
4. For $i = 1, \ldots, m$, apply Lemma 3.4 to f_i to find $g_i \in \mathbb{F}(y)$, $a_i, b_i \in \mathbb{F}[y]$ and $k_i \in \mathbb{N}$ with $\deg_y(a_i) < k_i \deg_y(d_i)$ such that (17) holds.
5. Set $g = \sum_{i=1}^m g_i, h = \sum_{i=1}^m a_i/d_i^{k_i}, b = \sum_{i=1}^m b_i$, and return.

Example 3.6. *Assume that σ_y is the q-shift operator. Let $K = -qy + 1$, which is σ_y-reduced, and let*

$$f = -\frac{q(q-1)y}{(qy-1)(q^2y-1)},$$

whose denominator has no nontrivial σ_y-special factors, and admits a strong σ_y-factorization $(qy-1)(q^2y-1) = d^\alpha$ with $d = -q^2y + 1$ and $\alpha = \sigma_y^{-1} + 1$. Applying the normal reduction to f with respect to K yields

$$f = \Delta_K\Big(\frac{1}{qy-1}\Big) + \frac{-1/q}{d} + \frac{1/q}{v},$$

where $v = 1$. Since the second summand is nonzero, by Theorem 3.5, $f \notin \operatorname{im}(\Delta_K)$.

3.2 Special reduction

We consider in this subsection the special part of a rational function in $\mathbb{F}(y)$, which, by Lemma 2.7, will be always zero in the usual shift case. Thus we merely need to address the q-shift case, in which the special part is a Laurent polynomial in $\mathbb{F}[y, y^{-1}]$, again by Lemma 2.7. For this purpose, we require the notion of σ_y-standard rational functions.

Definition 3.7. *A rational function in $\mathbb{F}(y)$ with numerator u and denominator v is said to be* σ_y-standard *if it is σ_y-reduced, and additionally in the q-shift case, $u(0)q^\ell - v(0) \neq 0$ for any negative integer ℓ.*

Theorem 3.8. *Assume that σ_y is the q-shift operator. Let $K \in \mathbb{F}(y)$ be σ_y-standard with denominator v, and let $f \in \mathbb{F}(y)$ be a proper rational function whose denominator is σ_y-special. Then there exist $g \in \mathbb{F}[y^{-1}]$ and $b \in \mathbb{F}[y]$ such that*

$$f = \Delta_K(g) + \frac{b}{v}. \tag{18}$$

Proof. Since σ_y is the q-shift operator and f is proper with σ_y-special denominator, it follows from Lemma 2.7 that $f = a/y^k$ for some $k \in \mathbb{N}$ and $a \in \mathbb{F}[y]$ with $\deg_y(a) < k$. Thus $vf = va/y^k$ is a Laurent polynomial in $\mathbb{F}[y, y^{-1}]$. Let $m = \operatorname{tdeg}(vf)$. If $m \geq 0$, then $vf \in \mathbb{F}[y]$ and thus letting $g = 0$ and $b = vf$ concludes the proof. Now assume that $m < 0$, and write $K = u/v$, where $u \in \mathbb{F}[y]$ with $\gcd(u, v) = 1$. Since K is σ_y-standard, $u(0)q^m - v(0) \neq 0$. Define

$$g_0 = \frac{cy^m}{u(0)q^m - v(0)},$$

where c is the coefficient of y^m in vf. Then $g_0 \in \mathbb{F}[y^{-1}]$ and

$$vf - (u\sigma_y(g_0) - vg_0) = vf - (cy^m + \text{higher terms in } y),$$

which is again a Laurent polynomial in $\mathbb{F}[y, y^{-1}]$ but with tail degree greater than m, as opposed to m for the initial Laurent polynomial vf. Thus, repeating this process at most $(-m)$ times yields a Laurent polynomial in $\mathbb{F}[y, y^{-1}]$ with tail degree at least 0, that is, a polynomial in $\mathbb{F}[y]$. In other words, we can find $g \in \mathbb{F}[y^{-1}]$ and $b \in \mathbb{F}[y]$ such that $vf = u\sigma_y(g) - vg + b$, giving

$$f = \frac{vf}{v} = \frac{u\sigma_y(g) - vg + b}{v} = \Delta_K(g) + \frac{b}{v}.$$

□

We now turn the above proof into the following algorithm.

SpecialReduction. Assume that σ_y is the q-shift operator. Given a σ_y-standard rational function $K \in \mathbb{F}(y)$ with numerator u and denominator v, and a proper rational function $f \in \mathbb{F}(y)$ whose denominator is σ_y-special, compute a Laurent polynomial $g \in \mathbb{F}[y^{-1}]$ and a polynomial $b \in \mathbb{F}[y]$ such that (18) holds.

1. Set $g = 0$, $b = vf$ and $m = \operatorname{tdeg}_y(b)$.
2. While $m < 0$ do
   - 2.1 Set $g_0 = cy^m/(u(0)q^m - v(0))$, where c is the coefficient of y^m in b.
   - 2.2 Update g to be $g + g_0$, b to be $b - (u\sigma_y(g_0) - vg_0)$, and m to be $\operatorname{tdeg}_y(b)$.
3. Return g and b.

Example 3.9. *Assume that σ_y is the q-shift operator. Let $K = -qy + 1$ and*

$$f = \frac{(q-1)(q^2-1)}{y^2},$$

whose denominator is σ_y-special. It is readily seen from definition that K is σ_y-standard. Applying the special reduction to f with respect to K yields

$$f = \Delta_K\Big(\frac{q^2(y - q + 1)}{y^2}\Big) + \frac{q^2}{v},$$

where $v = 1$.

3.3 Polynomial reduction

To deal with the remaining polynomial part, we present in this subsection a polynomial reduction which reduces the input polynomial into one lying in a finite-dimensional linear subspace over $\mathbb{F}$. This reduction was first presented in [11], and later extended in various ways [23, 24, 21, 27, 18].

Let $K \in \mathbb{F}(y)$ be σ_y-standard with numerator u and denominator v. We define an $\mathbb{F}$-linear map ϕ_K from $\mathbb{F}[y]$ to itself by sending p to $u\sigma_y(p) - vp$ for all $p \in \mathbb{F}[y]$, and call it the *map for polynomial reduction with respect to* K. Then the image of ϕ_K, denoted by $\operatorname{im}(\phi_K)$, is an $\mathbb{F}$-linear subspace of $\mathbb{F}[y]$. We denote by $\operatorname{im}(\phi_K)^\top$ the $\mathbb{F}$-linear subspace of $\mathbb{F}[y]$ spanned by monomials in y whose y-degrees are distinct from those of all polynomials in $\operatorname{im}(\phi_K)$, that is,

$$\operatorname{im}(\phi_K)^\top = \operatorname{span}_{\mathbb{F}} \left\{ y^d \mid d \in \mathbb{N} \text{ and } d \neq \deg_y(p) \text{ for all } p \in \operatorname{im}(\phi_K) \right\}.$$

Following the proof of [23, Lemma 4.1] verbatim, we obtain that $\mathbb{F}[y] = \operatorname{im}(\phi_K) \oplus \operatorname{im}(\phi_K)^\top$. Thus we call $\operatorname{im}(\phi_K)^\top$ the *standard complement of* $\operatorname{im}(\phi_K)$. Elements in a standard complement enjoy an important property, which will be useful in determining the σ_y-summability of σ_y-hypergeometric terms.

Lemma 3.10. *Let $K \in \mathbb{F}(y)$ be σ_y-standard with denominator v. If $p \in \operatorname{im}(\phi_K)^\top$ and $p/v \in \operatorname{im}(\Delta_K)$, then p is equal to zero.*

Proof. Write $K = u/v$, where $u \in \mathbb{F}[y]$ with $\gcd(u, v) = 1$. Assume that $p \in \operatorname{im}(\phi_K)^\top$ and $p/v \in \operatorname{im}(\Delta_K)$. Then there exists $g \in \mathbb{F}(y)$ such that $p/v = K\sigma_y(g) - g$, or equivalently,

$$p = u\sigma_y(g) - vg \in \mathbb{F}[y]. \tag{19}$$

It suffices to show that g is a polynomial in $\mathbb{F}[y]$, because then $p \in \operatorname{im}(\phi_K) \cap \operatorname{im}(\phi_K)^\top = \{0\}$ and thus $p = 0$. Suppose that $g \notin \mathbb{F}[y]$. Then its denominator d has a monic irreducible factor $a \in \mathbb{F}[y]$, which is either σ_y-normal or σ_y-special. Assume that a is σ_y-normal and let $\alpha \in \mathbb{N}[\sigma_y, \sigma_y^{-1}]$ be the σ_y-exponent of a in d with tail and head degrees m and n, respectively. Then $\sigma_y^m(a)$ is a factor of d but not a factor of $\sigma_y(d)$. It follows from (19) that $\sigma_y^m(a)$ divides v. Similarly, since $\sigma_y^{n+1}(a)$ is a factor of $\sigma_y(d)$ but not a factor of d, we have $\sigma_y^{n+1}(a)$ divides u by (19), which contradicts with the condition that K is σ_y-reduced. Thus a must be σ_y-special. Since a is irreducible, it does not belong to $\mathbb{F}$. Then it follows from Lemma 2.7 that σ_y is the q-shift operator and $a = y$. Thus $d = y^k$ for some $k \in \mathbb{Z}^+$ and $g = b/y^k$ for some $b \in \mathbb{F}[y]$ with $y \nmid b$. By (19), we get $y^k p = uq^{-k}\sigma_y(b) - vb$. Since σ_y is the q-shift operator and $k > 0$, letting $y = 0$ on the both sides of the above equation yields that $(u(0)q^{-k} - v(0))b(0) = 0$. Since $y \nmid b$, we have $b(0) \neq 0$ and thus $u(0)q^{-k} - v(0) = 0$. Note that k is positive. So we have derived a contradiction with the assumption that K is σ_y-standard. Therefore, $g \in \mathbb{F}[y]$ and then $p = 0$. □

Since $\mathbb{F}[y] = \operatorname{im}(\phi_K) \oplus \operatorname{im}(\phi_K)^\top$, a polynomial $p \in \mathbb{F}[y]$ can be uniquely decomposed as $p = p_1 + p_2$ with $p_1 \in \operatorname{im}(\phi_K)$ and $p_2 \in \operatorname{im}(\phi_K)^\top$. We will see shortly that such a decomposition can be easily computed using echelon bases for $\operatorname{im}(\phi_K)$ and $\operatorname{im}(\phi_K)^\top$. By an *echelon basis*, we mean an $\mathbb{F}$-basis in which different elements have distinct y-degrees. In order to obtain such a basis, we start by finding an ordinary $\mathbb{F}$-basis of $\operatorname{im}(\phi_K)$.

Lemma 3.11. *Let $K \in \mathbb{F}(y)$ be σ_y-standard. Then the following assertions hold.*

(i) The map ϕ_K is injective if K is unequal to one in the usual shift case or it is not a power of q in the q-shift case.

(ii) The set $\{\phi_K(y^i) \mid i \in \mathbb{N}\} \setminus \{0\}$ is an $\mathbb{F}$-basis for $\operatorname{im}(\phi_K)$.

Proof. Write $K = u/v$ with $u, v \in \mathbb{F}[y]$ and $\gcd(u, v) = 1$.

(i) Assume that K is unequal to one in the usual shift case or it is not a power of q in the q-shift case. Suppose that there exists a nonzero polynomial $p \in \mathbb{F}[y]$ such that $\phi_K(p) = 0$. Then $K = \sigma_y(1/p)/(1/p)$, a contradiction with Proposition 2.8, since K is σ_y-standard and thus σ_y-reduced. Therefore, $\phi_K(p) \neq 0$ for all $p \in \mathbb{F}[y] \setminus \{0\}$ and then the map ϕ_K is injective.

(ii) Let $\Lambda = \{\phi_K(y^i) \mid i \in \mathbb{N}\}$. It suffices to show the assertion when K is one in the usual shift case or K is a power of q in the q-shift case, for, otherwise, by part (i), the map ϕ_K is injective and thus $\Lambda \setminus \{0\} = \Lambda$ is an $\mathbb{F}$-basis for $\operatorname{im}(\phi_K)$.

In the usual shift case, namely the case when σ_y is the usual shift operator, assume that $K = 1$. Then we can take $u = v = 1$. It follows that $\phi_K(1) = 0$ and

$$\phi_K(y^i) = u(y+1)^i - vy^i = iy^{i-1} + (\text{lower terms in } y) \neq 0 \quad \text{for all } i \in \mathbb{Z}^+.$$

Thus $\Lambda \setminus \{0\} = \{\phi_K(y^i) \mid i \in \mathbb{Z}^+\}$, which is clearly an $\mathbb{F}$-basis for $\operatorname{im}(\phi_K)$.

In the q-shift case, namely the case when σ_y is the q-shift operator, assume that K is a power of q. Since K is σ_y-standard, we have $K = q^{-k}$ for some $k \in \mathbb{N}$. So we can take $u = q^{-k}$ and $v = 1$. It follows that $\phi_K(y^k) = 0$ and

$$\phi_K(y^i) = (q^{-k+i} - 1)y^i \neq 0 \quad \text{for all } i \in \mathbb{N} \setminus \{k\}.$$

Thus $\Lambda \setminus \{0\} = \{\phi_K(y^i) \mid i \in \mathbb{N} \setminus \{k\}\}$, which is again an $\mathbb{F}$-basis for $\operatorname{im}(\phi_K)$.

In summary, the set $\{\phi_K(y^i) \mid i \in \mathbb{N}\} \setminus \{0\}$ is always an $\mathbb{F}$-basis for $\operatorname{im}(\phi_K)$. □

We now make a case distinction to demonstrate how to construct an echelon basis for $\operatorname{im}(\phi_K)$, along with one for $\operatorname{im}(\phi_K)^\top$, from the ordinary $\mathbb{F}$-basis $\{\phi_K(y^i) \mid i \in \mathbb{N}\} \setminus \{0\}$. This distinction is slightly different from the one in [23, §4.2] for the usual shift case and that in [27, §5] for the q-shift case, in the sense that it also includes cases related to rational σ_y-hypergeometric terms.

Let $K \in \mathbb{F}(y)$ be σ_y-standard with numerator u and denominator v. Set

$$u = \sum_{i=0}^{d} u_i y^i \quad \text{and} \quad v = \sum_{i=0}^{d} v_i y^i,$$

where $d = \max\{\deg_y(u), \deg_y(v)\}$ and $u_i, v_i \in \mathbb{F}$ for all $i = 0, \ldots, d$. Note that u_d and v_d cannot be both zero.

Case 1. σ_y is the usual shift operator. Then

$$\begin{aligned} \phi_K(y^i) &= u(y+1)^i - vy^i = u((y+1)^i - y^i) + (u - v)y^i \\ &= (u_d - v_d)y^{d+i} + (iu_d + u_{d-1} - v_{d-1})y^{d+i-1} + (\text{lower terms in } y) \end{aligned} \tag{20}$$

for all $i \in \mathbb{N}$.

Case 1.1. $u_d - v_d \neq 0$. Then by (20), $\deg_y(\phi_K(y^i)) = d + i$ for all $i \in \mathbb{N}$. This implies that the images of different powers of y under ϕ_K have distinct y-degrees, and thus form an echelon basis for $\operatorname{im}(\phi_K)$. It follows that $\operatorname{im}(\phi_K)^\top$ has an echelon basis $\{1, y, \ldots, y^{d-1}\}$ and its dimension is equal to d.

Case 1.2. $u_d - v_d = 0$ and $d = 0$. Then $u_d = v_d \neq 0$ and $K = 1$. By (20), $\phi_K(1) = 0$ and $\deg_y(\phi_K(y^i)) = i - 1$ for all $i \in \mathbb{Z}^+$, implying that $\{\phi_K(y^i) \mid i \in \mathbb{Z}^+\}$ is an echelon basis for $\operatorname{im}(\phi_K)$ and thus $\operatorname{im}(\phi_K)^\top = \{0\}$, that is, $\dim(\operatorname{im}(\phi_K)^\top) = 0$.

Case 1.3. $u_d - v_d = 0$, $d > 0$ and $iu_d + u_{d-1} - v_{d-1} \neq 0$ for all $i \in \mathbb{N}$. Then by (20), $\deg_y(\phi_K(y^i)) = d + i - 1$ for all $i \in \mathbb{N}$. Similar to Case 1.1, we see that $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$ is an echelon basis for $\operatorname{im}(\phi_K)$. It follows that $\operatorname{im}(\phi_K)^\top$ has an echelon basis $\{1, y, \ldots, y^{d-2}\}$ and its dimension is equal to $d - 1$.

Case 1.4. $u_d - v_d = 0$, $d > 0$ and $ku_d + u_{d-1} - v_{d-1} = 0$ for some $k \in \mathbb{N}$. Note that $u_d = v_d \neq 0$. So the integer $k = (v_{d-1} - u_{d-1})/u_d$ is unique. Then by (20), $\deg_y(\phi_K(y^i)) = d + i - 1$ for all $i \in \mathbb{N}$ with $i \neq k$, and $\deg_y(\phi_K(y^k)) < d + k - 1$. Since $d > 0$, we have $K \neq 1$. By Lemma 3.11, $\phi_K(y^k) \neq 0$ and $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$ is an $\mathbb{F}$-basis for $\mathrm{im}(\phi_K)$. Eliminating $y^{d+k-2}, y^{d+k-3}, \ldots, y^{d-1}$ from $\phi_K(y^k)$ successively by the elements $\phi_K(y^{k-1}), \phi_K(y^{k-2}), \ldots, \phi_K(y^0)$, we will obtain a polynomial $r \in \mathbb{F}[y]$ with $\deg_y(r) < d - 1$. Since $\phi_K(y^0), \ldots, \phi_K(y^{k-1}), \phi_K(y^k)$ are linearly independent over $\mathbb{F}$, the polynomial r is nonzero. So $\{\phi_K(y^i) \mid i \in \mathbb{N} \text{ with } i \neq k\} \cup \{r\}$ is an echelon basis for $\mathrm{im}(\phi_K)$. It follows that $\mathrm{im}(\phi_K)^\top$ has an echelon basis

$$\{1, y, \ldots, y^{\deg_y(r)-1}, y^{\deg_y(r)+1}, \ldots, y^{d-2}, y^{d+k-1}\}$$

and its dimension is equal to $d - 1$.

Case 2. σ_y is the q-shift operator. Then

$$\phi_K(y^i) = uq^i y^i - vy^i = (u_d q^i - v_d)y^{d+i} + (u_{d-1}q^i - v_{d-1})y^{d+i-1} + \cdots + (u_0 q^i - v_0)y^i \tag{21}$$

for all $i \in \mathbb{N}$.

Case 2.1. $u_d q^i - v_d \neq 0$ for all $i \in \mathbb{N}$. Then by (21), $\phi_K(y^i) \neq 0$ and $\deg_y(\phi_K(y^i)) = d + i$ for all $i \in \mathbb{N}$. Thus $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$ is an echelon basis for $\mathrm{im}(\phi_K)$. It follows that $\mathrm{im}(\phi_K)^\top$ has an echelon basis $\{1, y, \ldots, y^{d-1}\}$ and its dimension is equal to d.

Case 2.2. $u_d q^k - v_d = 0$ for some $k \in \mathbb{N}$ and $d = 0$. Then $u_d q^k = v_d \neq 0$ and $K = q^{-k}$. By (21), $\phi_K(y^k) = 0$ and $\deg_y(\phi_K(y^i)) = i$ for all $i \in \mathbb{N}$ with $i \neq k$. Thus $\{\phi_K(y^i) \mid i \in \mathbb{N} \setminus \{k\}\}$ is an echelon basis for $\mathrm{im}(\phi_K)$. It follows that $\mathrm{im}(\phi_K)^\top$ is a one-dimensional subspace of $\mathbb{F}[y]$ spanned by $\{y^k\}$.

Case 2.3. $u_d q^k - v_d = 0$ for some $k \in \mathbb{N}$ and $d > 0$. The integer k is unique, because q is neither zero nor a root of unity. Then by (21), $\deg_y(\phi_K(y^i)) = d + i$ for all $i \in \mathbb{N}$ with $i \neq k$, and $\deg_y(\phi_K(y^k)) < d + k$. Since $d > 0$, we know that K cannot be a power of q. It then follows from Lemma 3.11 that $\phi_K(y^k) \neq 0$ and $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$ is an $\mathbb{F}$-basis for $\mathrm{im}(\phi_K)$. Similar to Case 1.4, we can successively eliminate $y^{d+k-1}, y^{d+k-2}, \ldots, y^d$ from $\phi_K(y^k)$ by $\phi_K(y^{k-1}), \phi_K(y^{k-2}), \ldots, \phi_K(y^0)$ and obtain a polynomial $r \in \mathbb{F}[y]$ with $\deg_y(r) < d$. Since $\phi_K(y^0), \ldots, \phi_K(y^{k-1}), \phi_K(y^k)$ are linearly independent over $\mathbb{F}$, we have $r \neq 0$. Thus $\{\phi_K(y^i) \mid i \in \mathbb{N} \text{ with } i \neq k\} \cup \{r\}$ is an echelon basis for $\mathrm{im}(\phi_K)$. It follows that $\mathrm{im}(\phi_K)^\top$ has an echelon basis

$$\{1, y, \ldots, y^{\deg_y(r)-1}, y^{\deg_y(r)+1}, \ldots, y^{d-1}, y^{d+k}\}$$

and its dimension is equal to d.

The above case distinction leads to an interesting consequence, which tells us that the standard complement has a finite dimension, implying that all polynomials therein are "sparse".

Proposition 3.12. *Let $K \in \mathbb{F}(y)$ be σ_y-standard with numerator u and denominator v. The standard complement of $\mathrm{im}(\phi_K)$ is of dimension*

$$\max\{\deg_y(u), \deg_y(v)\} + \begin{cases} -[\![0 \leq \deg_y(u - v) \leq \deg_y(u) - 1]\!] & \text{in the usual shift case,} \\ [\![K \text{ is a nonpositive power of } q]\!] & \text{in the q-shift case,} \end{cases} \tag{22}$$

where the notation $[\![\cdot]\!]$ equals 1 if the argument is true and 0 otherwise.

Example 3.13. *Assume that σ_y is the q-shift operator. Let $K = -qy + 1$, which is σ_y-standard. According to Case 2.1, $\mathrm{im}(\phi_K)$ has an echelon basis $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$. Thus $\mathrm{im}(\phi_K)^\top$ has a basis $\{1\}$ and its dimension is one.*

With echelon bases at hand, we are able to project a polynomial onto $\mathrm{im}(\phi_K)$ and $\mathrm{im}(\phi_K)^\top$, respectively. The main process is summarized below.

PolynomialReduction. Given a σ_y-standard rational function $K \in \mathbb{F}(y)$, and a polynomial $b \in \mathbb{F}[y]$, compute two polynomials $a, p \in \mathbb{F}[y]$ with $p \in \mathrm{im}(\phi_K)^\top$ such that $b = \phi_K(a) + p$.

1. If $b = 0$ then set $a = 0$ and $p = 0$; return.
2. Find polynomials $g_1, \ldots, g_m \in \mathbb{F}[y]$ such that the set $\{\phi_K(g_1), \ldots, \phi_K(g_m)\}$ consists of all polynomials in an echelon basis of $\mathrm{im}(\phi_K)$ whose y-degrees are no more than $\deg_y(b)$, and
$$0 \le \deg_y(\phi_K(g_1)) < \cdots < \deg_y(\phi_K(g_m)) \le \deg_y(b).$$
3. For $i = m, m-1, \ldots, 1$, perform linear elimination to find $c_m, c_{m-1}, \ldots, c_1 \in \mathbb{F}$ such that
$$b - \sum_{i=1}^{m} c_i \phi_K(g_i) \in \mathrm{im}(\phi_K)^\top.$$
4. Set $a = \sum_{i=1}^{m} c_i g_i$ and $p = b - \sum_{i=1}^{m} c_i \phi_K(g_i)$, and return.

3.4 Remainders of rational functions

Incorporating the normal, the special and the polynomial reduction, we are able to further reduce the shell to a "minimal normal form". The following definition formalizes such a form.

Definition 3.14. *Let $K \in \mathbb{F}(y)$ be σ_y-standard with denominator v, and let $f \in \mathbb{F}(y)$. Another rational function r in $\mathbb{F}(y)$ is called a σ_y-*remainder *of f with respect to K if $f - r \in \mathrm{im}(\Delta_K)$ and r can be written in the form*
$$r = h + \frac{p}{v}, \tag{23}$$
where $h \in \mathbb{F}(y)$ is proper with denominator being σ_y-normal and strongly coprime with K, and $p \in \mathrm{im}(\phi_K)^\top$. For brevity, we just say that r is a σ_y-remainder with respect to K if f is clear from the context. In addition, we call the denominator of h the significant denominator *of r.*

The notion of significant denominators is well-defined, since the denominator of h in (23) is strongly coprime with K and thus is coprime with v.

Remainders describe the "minimum" distance of a rational function in $\mathbb{F}(y)$ from the $\mathbb{F}$-linear subspace $\mathrm{im}(\Delta_K)$, and help us decide the σ_y-summability.

Proposition 3.15. *Let $K \in \mathbb{F}(y)$ be σ_y-standard with numerator u and denominator v, and let r be a σ_y-remainder with respect to K of the form (23). Then the following assertions hold.*

(i) The total number of nonzero terms in p is no more than the number given by (22).

(ii) If there exists $\tilde{r} \in \mathbb{F}(y)$ such that $r - \tilde{r} \in \mathrm{im}(\Delta_K)$, then by writing $\tilde{r}$ in the form
$$\tilde{r} = \tilde{h} + \frac{\tilde{p}}{v} \tag{24}$$
for some $\tilde{h} \in \mathbb{F}(y)$ and $\tilde{p} \in \mathbb{F}[y]$, we have the y-degree of the denominator of h is no more than that of $\tilde{h}$.

(iii) $r \in \mathrm{im}(\Delta_K)$ if and only if $r = 0$.

Proof. (i) Since $p \in \operatorname{im}(\phi_K)^\top$, the assertion immediately follows by Proposition 3.12.
(ii) Assume that there exists $\tilde{r} \in \mathbb{F}(y)$ such that $r - \tilde{r} \in \operatorname{im}(\Delta_K)$ and write $\tilde{r}$ in the form (24). Then by (23), we obtain

$$h - \tilde{h} + \frac{p - \tilde{p}}{v} \in \operatorname{im}(\Delta_K).$$

The assertion is thus evident by Lemma 3.2.
(iii) The sufficiency is clear. For the necessity, assume that $r \in \operatorname{im}(\Delta_K)$, that is, $h + p/v \in \operatorname{im}(\Delta_K)$. We see from Theorem 3.5 that h is actually the zero polynomial. It then follows that $p/v \in \operatorname{im}(\Delta_K)$. Since K is σ_y-standard and $p \in \operatorname{im}(\phi_K)^\top$, we have $p = 0$ by Lemma 3.10, and thus $r = 0$. □

For describing our reduction algorithm, it remains to show that every nonzero rational function in $\mathbb{F}(y)$ has a σ_y-standard kernel, and it is not hard to construct one.

Proposition 3.16. *Let f be a nonzero rational function in $\mathbb{F}(y)$. Then f has a σ_y-standard kernel K. Moreover, there exists $S \in \mathbb{F}(y)$ whose denominator does not have any nontrivial σ_y-special factors such that (K, S) is an RNF of f.*

Proof. The assertions are evident in the usual shift case, because in this case, σ_y-standard rational functions in $\mathbb{F}(y)$ are exactly σ_y-reduced ones, and all σ_y-special polynomials in $\mathbb{F}[y]$ belong to $\mathbb{F}$ by Lemma 2.7.

Now assume that σ_y is the q-shift operator. Let $f \in \mathbb{F}(y) \setminus \{0\}$ and $(\tilde{K}, \tilde{S})$ be an RNF of f. Then $\tilde{K}$ is σ_y-reduced. Notice that $\sigma_y(y^\ell)/y^\ell = q^\ell$ for all $\ell \in \mathbb{Z}$. So we may assume without loss of generality that $\tilde{S}$ is σ_y-monic. Write $\tilde{K} = u/v$ with $u, v \in \mathbb{F}[y]$ and $\gcd(u, v) = 1$. Notice that the constant terms $u(0)$ and $v(0)$ cannot be both zero. If one of them is zero, or neither is zero and meanwhile $u(0)/v(0)$ is not a positive power of q, then $u(0)q^\ell - v(0) \neq 0$ for any negative integer ℓ as q is nonzero, implying that $\tilde{K}$ is already σ_y-standard and $(\tilde{K}, \tilde{S})$ gives a desired RNF of f. Suppose that $u(0), v(0)$ are both nonzero and $u(0)/v(0) = q^m$ for some $m \in \mathbb{Z}^+$. Define $K = q^{-m}\tilde{K}$ and $S = y^m\tilde{S}$. Since $(\tilde{K}, \tilde{S})$ is an RNF of f, it follows that

$$f = \tilde{K}\frac{\sigma_y(\tilde{S})}{\tilde{S}} = q^{-m}\tilde{K}\frac{\sigma_y(y^m\tilde{S})}{y^m\tilde{S}} = K\frac{\sigma_y(S)}{S}.$$

Since $\tilde{K}$ is σ_y-reduced, so is K and thus (K, S) is an RNF of f. Since $K = q^{-m}\tilde{K}$, it has numerator $q^{-m}u$ and denominator v. Notice that $u(0)/v(0) = q^m$ and q is not a root of unity. So $q^{-m}u(0)q^\ell - v(0) = v(0)(q^\ell - 1) \neq 0$ for any negative integer ℓ. It follows that K is σ_y-standard. It remains to show that the denominator of S has no nontrivial σ_y-special factors, which, by Lemma 2.7, is equivalent to verify that y does not divide the denominator of S. This follows by the observation that $S = y^m\tilde{S}$, $m > 0$ and $\tilde{S}$ is σ_y-monic. □

With everything in place, we now present our reduction algorithm, which determines the σ_y-summability of a σ_y-hypergeometric term without solving any auxiliary difference equations explicitly.

HypergeomReduction. Given a σ_y-hypergeometric term T, compute a σ_y-hypergeometric term H whose σ_y-quotient K is σ_y-standard, and two rational functions $g, r \in \mathbb{F}(y)$ such that

$$T = \Delta_y(gH) + rH, \tag{25}$$

and r is a σ_y-remainder with respect to K.

1. Find a σ_y-standard kernel K and a corresponding shell S of $\sigma_y(T)/T$, and set $H = T/S$.

2. Compute the canonical representation

$$S = f_p + f_s + f_n, \tag{26}$$

where f_p, f_s and f_n are the polynomial, special and normal parts of S, respectively.

3. Apply **NormalReduction** to f_n with respect to K to find $g, h \in \mathbb{F}(y)$ and $b \in \mathbb{F}[y]$ such that $f_n = \Delta_K(g) + h + b/v$, and h is proper whose denominator is σ_y-normal and strongly coprime with K.

4. If $f_s \neq 0$ then apply **SpecialReduction** to f_s with respect to K to find $g_s \in \mathbb{F}[y^{-1}]$ and $b_s \in \mathbb{F}[y]$ such that $f_s = \Delta_K(g_s) + b_s/v$, and update g to be $g + g_s$ and b to be $b + b_s$.

5. Apply **PolynomialReduction** to $vf_p + b$ with respect to K to find $a \in \mathbb{F}[y]$ and $p \in \mathrm{im}(\phi_K)^\top$ such that $vf_p + b = \phi_K(a) + p$.

6. Update g to be $a + g$ and set $r = h + p/v$, and return H, g, r.

Remark 3.17. *If the shell found in step 1 of the above algorithm is further chosen to be one whose denominator contains no nontrivial σ_y-special factors (cf. Proposition 3.16), then step 4 in the above algorithm can be completely skipped.*

Theorem 3.18. *For a σ_y-hypergeometric term T, the algorithm* **HypergeomReduction** *computes a σ_y-hypergeometric term H whose σ_y-quotient K is σ_y-standard, a rational function $g \in \mathbb{F}(y)$ and a σ_y-remainder r with respect to K such that (25) holds. Moreover, T is σ_y-summable if and only if $r = 0$.*

Proof. The correctness of step 1 is guaranteed by Proposition 3.16. Since $S \in \mathbb{F}(y)$, the canonical representation (26) holds. Applying **NormalReduction** to f_n and **SpecialReduction** to f_s if $f_s \neq 0$ with respect to K, we obtain, after step 4, that

$$f_n + f_s = \Delta_K(g) + h + \frac{b}{v},$$

which, together with (26), leads to

$$S = \Delta_K(g) + h + \frac{vf_p + b}{v}. \tag{27}$$

The algorithm **PolynomialReduction** in step 5 then computes the decomposition

$$vf_p + b = \phi_K(a) + p = u\sigma_y(a) - va + p.$$

Substituting this into (27), we see that

$$S = \Delta_K(g) + h + K\sigma_y(a) - a + \frac{p}{v} = \Delta_K(a + g) + h + \frac{p}{v}.$$

After renaming the symbols in step 6 and multiplying both sides by H, we get (25) since $T = SH$ and $K = \sigma_y(H)/H$. Thus T is σ_y-summable if and only if rH is σ_y-summable, which happens if and only if $r \in \mathrm{im}(\Delta_K)$ and then, by Proposition 3.15 (iii), is equivalent to say that $r = 0$. □

Example 3.19. *Assume that σ_y is the usual shift operator. Let*

$$T = \frac{y^3 + 4y^2 + 2y - 2}{(y+1)(y+2)} y!.$$

Clearly, T is a hypergeometric term whose σ_y-quotient has a kernel $K = y+1$ and a corresponding shell

$$S = \frac{y^3 + 4y^2 + 2y - 2}{(y+1)(y+2)}.$$

Since K is σ_y-reduced, it is σ_y-standard by definition. Note that in the usual shift case, all σ_y-special polynomials in $\mathbb{F}[y]$ belong to $\mathbb{F}$. So we have obtained a σ_y-standard kernel K and a corresponding shell S whose denominator $(y+1)(y+2)$ has no nontrivial σ_y-special factors. Computing the canonical representation of S gives

$$S \;=\; \underbrace{y+1}_{f_p} \;+\; \underbrace{0}_{f_s} \;+\; \underbrace{\left(-\frac{3y+4}{(y+1)(y+2)}\right)}_{f_n}.$$

For the normal part f_n, its denominator admits a strong σ_y-factorization $(y+1)(y+2) = d^\alpha$ with $d = y+2$ and $\alpha = \sigma_y^{-1}+1$. Applying the normal reduction to f_n with respect to K yields

$$-\frac{3y+4}{(y+1)(y+2)} = \Delta_K\left(\frac{1}{y+1}\right) - \frac{1}{y+2} - \frac{1}{v},$$

where $v = 1$. Since $f_s = 0$, we skip the special reduction. Combining with the polynomial part f_p, we then use the polynomial reduction with respect to K to obtain $v(y+1)-1 = \phi_K(1)+0$. Thus

$$S = \Delta_K\left(\frac{y+2}{y+1}\right) - \frac{1}{y+2},$$

and consequently,

$$T = \Delta_y\left(\frac{y+2}{y+1}H\right) - \frac{1}{y+2}H,$$

where $H = T/S = y!$. So the term T is not σ_y-summable.

Example 3.20. *Assume that σ_y is the q-shift operator. Let*

$$T(k) = \frac{q^k(q^{2k+3} - q^{k+2} - q^{k+1} - q^2 + q + 1)}{(q^{k+1}-1)(q^{k+2}-1)}(q,q)_k.$$

Clearly, T is a q-hypergeometric term with $q^k = y$ and $\sigma_y(T(k)) = T(k+1)$. Then the σ_y-quotient of T has a kernel $\tilde{K} = -q(qy-1)$ and a corresponding shell

$$\tilde{S} = \frac{q^3y^2 - q^2y - qy - q^2 + q + 1}{(qy-1)(q^2y-1)}.$$

Notice that $\tilde{K}$ is not σ_y-standard by definition. Performing the standardization process as in the proof of Proposition 3.16, we obtain a σ_y-standard kernel $K = -qy+1$ and a corresponding shell $S = y\tilde{S}$, whose denominator $(qy-1)(q^2y-1)$ has no nontrivial σ_y-special factors. Computing the canonical representation of S gives

$$S = \underbrace{y}_{f_p} \;+\; \underbrace{0}_{f_s} \;+\; \underbrace{\left(-\frac{q(q-1)y}{(qy-1)(q^2y-1)}\right)}_{f_n}.$$

Applying the normal reduction as in Example 3.6, we decompose the normal part f_n as

$$-\frac{q(q-1)y}{(qy-1)(q^2y-1)} = \Delta_K\left(\frac{1}{qy-1}\right) + \frac{1}{q(q^2y-1)} + \frac{1/q}{v},$$

where $v = 1$. Since $f_s = 0$, we skip the special reduction. Combining with the polynomial part f_p, we then use the polynomial reduction with respect to K to obtain $v \cdot y + 1/q = \phi_K(-1/q) + 1/q$. Thus

$$S = \Delta_K\left(-\frac{qy-q-1}{q(qy-1)}\right) + \frac{qy}{q^2y-1},$$

and consequently,

$$T = \Delta_y\left(-\frac{q^{k+1}-q-1}{q(q^{k+1}-1)}H\right) + \frac{q^{k+1}}{q^{k+2}-1}H,$$

where $H = T/S = (q,q)_k$. So the term T is not σ_y-summable.

4 Sum of σ_y-remainders

In order to compute telescopers for σ_y-hypergeometric terms, we want to parameterize the reduction algorithm developed in the preceding section so as to reduce the problem to determining the linear dependency among certain σ_y-remainders. However, we are confronted with the same difficulty as mentioned in [23, §5] that the sum of two σ_y-remainders is not necessarily a σ_y-remainder. A complete obstruction preventing the linearity from being true is that the least common multiple of two σ_y-normal polynomials may not again be σ_y-normal. The idea used in [23, §5] to circumvent this obstruction can be literally extended to our general setting. For the sake of completeness and the convenience of later use, we present the idea in this section with some subtle adjustments.

Let d and e be two nonzero σ_y-normal polynomials in $\mathbb{F}[y]$. By polynomial factorization and dispersion computation (see [5]), one can decompose

$$e = \tilde{e}\, d_1^{k_1\sigma_y^{\ell_1}} \cdots d_m^{k_m\sigma_y^{\ell_m}}, \tag{28}$$

where $\tilde{e} \in \mathbb{F}[y]$ is σ_y-coprime with d, $d_1, \ldots, d_m \in \mathbb{F}[y]$ are pairwise distinct and σ_y-monic irreducible factors of d, $\ell_1, \ldots, \ell_m$ are nonzero integers, and $k_1, \ldots, k_m \in \mathbb{Z}^+$ are multiplicities of the factors $\sigma_y^{\ell_1}(d_1), \ldots, \sigma_y^{\ell_m}(d_m)$ in e, respectively. Note that such a decomposition is unique up to the order of factors. Following [23], we refer to (28) as the *σ_y-coprime decomposition* of e with respect to d.

Theorem 4.1. *Let $K \in \mathbb{F}(y)$ be σ_y-standard, and let r, s be two σ_y-remainders with respect to K. Then there exists a σ_y-remainder t with respect to K such that $s - t \in \mathrm{im}(\Delta_K)$ and $\lambda r + \mu t$ for all $\lambda, \mu \in \mathbb{F}$ is a σ_y-remainder with respect to K.*

Proof. Let d and e be significant denominators of r and s, respectively. Then d and e are both σ_y-normal and strongly coprime with K. Let (28) be the σ_y-coprime decomposition of e with respect to d. Since d and e are both σ_y-normal, the factors $\tilde{e}, \sigma_y^{\ell_1}(d_1), \ldots, \sigma_y^{\ell_m}(d_m)$ are pairwise coprime. Then s can be decomposed as

$$s = \sum_{i=1}^{m} s_i + \frac{\tilde{a}}{\tilde{e}} + \frac{b}{v}, \tag{29}$$

where $s_i \in \mathbb{F}(y)$ is a nonzero proper rational function with denominator $d_i^{k_i\sigma_y^{\ell_i}}$ for $i = 1, \ldots, m$, $\tilde{a} \in \mathbb{F}[y]$ with $\deg_y(\tilde{a}) < \deg_y(\tilde{e})$, $b \in \mathrm{im}(\phi_K)^\top$ and v is the denominator of K. For each $i = 1, \ldots, m$, applying Lemma 3.3 to s_i delivers

$$s_i = \Delta_K(g_i) + \frac{a_i}{d_i^{k_i}} + \frac{b_i}{v}, \tag{30}$$

where $g_i \in \mathbb{F}(y)$, $a_i, b_i \in \mathbb{F}[y]$ and $\deg_y(a_i) < k_i \deg_y(d_i)$. It then follows from (29) that

$$s = \Delta_K\Big(\sum_{i=1}^{m} g_i\Big) + \sum_{i=1}^{m} \frac{a_i}{d_i^{k_i}} + \frac{\tilde{a}}{\tilde{e}} + \frac{b + \sum_{i=1}^{m} b_i}{v}.$$

By polynomial reduction, we can find $a \in \mathbb{F}[y]$ and $p \in \mathrm{im}(\phi_K)^\top$ so that $\sum_{i=1}^{m} b_i = \phi_K(a) + p$. Let

$$h = \sum_{i=1}^{m} \frac{a_i}{d_i^{k_i}} + \frac{\tilde{a}}{\tilde{e}} \quad \text{and} \quad t = h + \frac{b+p}{v}.$$

Then $s = \Delta_K(\sum_{i=1}^{m} g_i + a) + t$ and thus $s - t \in \mathrm{im}(\Delta_K)$.

Notice that the denominator of h divides the polynomial $\tilde{e}\, d_1^{k_1} \cdots d_m^{k_m}$, which is σ_y-normal and strongly coprime with K since both d and e are σ_y-normal and strongly coprime with K, and is σ_y-coprime with d since $\tilde{e}$ is σ_y-coprime with d and $d_i \mid d$ for all $i = 1, \ldots, m$. Thus h is proper whose denominator is σ_y-normal, strongly coprime with K and σ_y-coprime with d. Since $b + p \in \text{im}(\phi_K)^\top$, we conclude that t is a σ_y-remainder with respect to K whose significant denominator is σ_y-coprime with d. It follows that the least common multiple of the significant denominators of t and r is σ_y-normal. Therefore, $\lambda r + \mu t$ for all $\lambda, \mu \in \mathbb{F}$ is a σ_y-remainder with respect to K. □

The proof of the above theorem contains an algorithm, which is outlined below.

RemainderLinearization. Given a σ_y-standard rational function $K \in \mathbb{F}(y)$, and two σ_y-remainders r, s with respect to K, compute a rational function $g \in \mathbb{F}(y)$ and another σ_y-remainder t with respect to K such that

$$s = \Delta_K(g) + t$$

and $\lambda r + \mu t$ for all $\lambda, \mu \in \mathbb{F}$ is a σ_y-remainder with respect to K.

1. Set d and e to be the significant denominators of r and s, respectively.
2. Compute the σ_y-coprime decomposition (28) of e with respect to d, and then decompose s into the form (29).
3. For $i = 1, \ldots, m$, apply Lemma 3.3 to s_i to find $g_i \in \mathbb{F}(y)$ and $a_i, b_i \in \mathbb{F}[y]$ with $\deg_y(a_i) < k_i \deg_y(d_i)$ such that (30) holds.
4. Apply **PolynomialReduction** to $\sum_{i=1}^m b_i$ to find $a \in \mathbb{F}[y]$ and $p \in \text{im}(\phi_K)^\top$ such that $\sum_{i=1}^m b_i = \phi_K(a) + p$.
5. Set
$$g = \sum_{i=1}^m g_i + a \quad \text{and} \quad t = \sum_{i=1}^m \frac{a_i}{d_i^{k_i}} + \frac{\tilde{a}}{\tilde{e}} + \frac{b + p}{v},$$
and return.

Example 4.2. *Assume that σ_y is the q-shift operator. Let $K = -qy + 1$, which is σ_y-standard, and let $r = qy/(q^2y - 1)$, $s = q^2y/(q^3y - 1)$. Then both r and s are σ_y-remainders with respect to K, but their sum is not one, because the denominator $(q^2y - 1)(q^3y - 1)$ is not σ_y-normal. Using the algorithm* **RemainderLinearization**, *we can find another σ_y-remainder $t = q^3y/(q^2 - 1)/(q^2y - 1)$ of s with respect to K such that $r + t = q(2q^2 - 1)y/(q^2 - 1)/(q^2y - 1)$, which is clearly a σ_y-remainder with respect to K.*

5 Creative telescoping via reduction

In this section, we will translate terminologies concerning univariate hypergeometric terms to bivariate ones and propose an algorithm for computing minimal telescopers as well as certificates in this bivariate setting.

Let $\mathbb{K}$ be a field of characteristic zero, and $\mathbb{K}(x, y)$ be the field of rational functions in x and y over $\mathbb{K}$. Let σ_x and σ_y be both either the usual shift operators with respect to x and y respectively defined by

$$\sigma_x(f(x, y)) = f(x + 1, y) \quad \text{and} \quad \sigma_y(f(x, y)) = f(x, y + 1),$$

or the q-shift operators with respect to x and y respectively defined by

$$\sigma_x(f(x, y)) = f(qx, y) \quad \text{and} \quad \sigma_y(f(x, y)) = f(x, qy)$$

for any $f \in \mathbb{K}(x, y)$, where $q \in \mathbb{K}$ is neither zero nor a root of unity. Similarly, we will refer to the former case as the usual shift case, and the latter one as the q-shift case. The pair $(\mathbb{K}(x, y), \{\sigma_x, \sigma_y\})$ forms a partial (q-)difference field. Let R be a *partial (q-)difference ring extension* of $(\mathbb{K}(x, y), \{\sigma_x, \sigma_y\})$, that is, R is a ring containing $\mathbb{K}(x, y)$ together with two distinguished endomorphisms σ_x and σ_y from R to itself, whose respective restrictions to $\mathbb{K}(x, y)$ agree with the two automorphisms defined earlier. In analogy with the univariate case in Section 2, an element $c \in R$ is called a *constant* if it is invariant under the applications of σ_x and σ_y. It is readily seen that all constants in R form a subring of R.

Definition 5.1. *An invertible element T of R is called a* (σ_x, σ_y)-hypergeometric term *if there exist $f, g \in \mathbb{K}(x, y)$ such that $\sigma_x(T) = fT$ and $\sigma_y(T) = gT$. We call f and g the σ_x- and σ_y-quotients of T, respectively.*

In the rest of this paper, let $\mathbb{F}$ be the field $\mathbb{K}(x)$ and $\mathbb{F}[S_x]$ be the ring of linear recurrence operators in x over $\mathbb{F}$, in which the commutation rule is that $S_x f = \sigma_x(f) S_x$ for all $f \in \mathbb{F}$. The application of an operator $L = \sum_{i=0}^{\rho} \ell_i S_x^i \in \mathbb{F}[S_x]$ to a (σ_x, σ_y)-hypergeometric term T is defined as

$$L(T) = \sum_{i=0}^{\rho} \ell_i \sigma_x^i(T).$$

Definition 5.2. *Let T be a (σ_x, σ_y)-hypergeometric term. A nonzero linear recurrence operator $L \in \mathbb{F}[S_x]$ is called a* telescoper *for T if there exists a (σ_x, σ_y)-hypergeometric term G such that*

$$L(T) = \Delta_y(G).$$

We call G a corresponding certificate *of L. The* order *of a telescoper for T is defined to be its degree in S_x.*

For (σ_x, σ_y)-hypergeometric terms, telescopers do not always exist. Existence criteria were provided by Abramov [1] for the usual shift case and by Chen et al. [26] for the q-shift case. In order to describe them concisely, we introduce the notion of integer-linear rational functions. Note that two polynomials a, b in $\mathbb{K}[x, y]$ are associates if and only if $a = c\,b$ for some $c \in \mathbb{K}$.

Definition 5.3. *An irreducible polynomial p in $\mathbb{K}[x, y]$ is said to be* integer-linear *(over $\mathbb{K}$) if there exist $m, n \in \mathbb{Z}$, not both zero, such that p and $\sigma_x^m \sigma_y^n(p)$ are associates. A polynomial in $\mathbb{K}[x, y]$ is said to be* integer-linear *(over $\mathbb{K}$) if all its irreducible factors over $\mathbb{K}$ are integer-linear. A rational function in $\mathbb{F}(y)$ is said to be* integer-linear *(over $\mathbb{K}$) if its denominator and numerator are both integer-linear.*

We refer to [29, 30] for algorithms determining the integer-linearity of rational functions. The following proposition provides an easy-to-use equivalent form for an integer-linear irreducible polynomial in $\mathbb{K}[x, y]$, which also justifies the "integer-linear" attribute in the name.

Proposition 5.4. *Let p be an irreducible polynomial in $\mathbb{K}[x, y]$.*

(i) In the usual shift case, p is integer-linear if and only if $p(x, y) = P(\lambda x + \mu y)$ for some $P(z) \in \mathbb{K}[z]$ and $\lambda, \mu \in \mathbb{Z}$ not both zero.

(ii) In the q-shift case, p is integer-linear if and only if $p(x, y) = x^\alpha y^\beta P(x^\lambda y^\mu)$ for some $P(z) \in \mathbb{K}[z]$ and $\alpha, \beta, \lambda, \mu \in \mathbb{Z}$ with λ, μ not both zero.

Proof. (i) We first consider the usual shift case, that is, the case when σ_x and σ_y are both usual shift operators. The sufficiency is clear since $\sigma_x^\mu \sigma_y^{-\lambda}(P(\lambda x + \mu y)) = P(\lambda x + \mu y)$ for any $P(z) \in \mathbb{K}[z]$ and $\lambda, \mu \in \mathbb{Z}$. Assume that p is integer-linear. Then there are $m, n \in \mathbb{Z}$, not both zero, such that p and $\sigma_x^m \sigma_y^n(p)$ are associates. Since σ_x, σ_y are usual shift operators, we have $p(x + m, y + n) = p(x, y)$. The assertion then follows from [34, Lemma 3.3].

(ii) In the q-shift case, that is, in the case when σ_x and σ_y are both q-shift operators, the sufficiency is clear since $\sigma_x^{\mu}\sigma_y^{-\lambda}(x^\alpha y^\beta P(x^\lambda y^\mu)) = q^{\alpha\mu-\beta\lambda}x^\alpha y^\beta P(x^\lambda y^\mu)$ for any $P(z) \in \mathbb{K}[z]$ and $\alpha, \beta, \lambda, \mu \in \mathbb{Z}$. For the necessity, we assume that p is integer-linear. Then p is an associate of $\sigma_x^m \sigma_y^n(p)$ for some $m, n \in \mathbb{Z}$ not both zero. Since p is irreducible over $\mathbb{K}$ and σ_x, σ_y are q-shift operators, it follows that $p(q^m x, q^n y) = c\,p(x, y)$ for some $c \in \mathbb{K}$. We now adapt the proof of [34, Lemma 3.3] into this case. Let $\ell = \gcd(m, n)$. Since m, n are not both zero, we have $\ell \neq 0$. Let $\lambda = -n/\ell$ and $\mu = m/\ell$. It is readily seen that $\gcd(\lambda, \mu) = 1$. By Bézout's relation, there exist $s, t \in \mathbb{Z}$ such that $s\lambda + t\mu = 1$. Define $h(x, y) = p(x^s y^\mu, x^t y^{-\lambda})$. Then $h \in \mathbb{K}[x, x^{-1}, y, y^{-1}] \subset \mathbb{F}(y)$ and

$$h(x, q^\ell y) = p(x^s(q^\ell y)^\mu, x^t(q^\ell y)^{-\lambda}) = p(q^m x^s y^\mu, q^n x^t y^{-\lambda}) = c\,p(x^s y^\mu, x^t y^{-\lambda}) = c\,h(x, y).$$

Since $\ell \neq 0$, we conclude from Lemma 2.6 (ii) that $h/y^k \in \mathbb{F}$ for some integer k. Notice that $h \in \mathbb{K}[x, x^{-1}, y, y^{-1}]$. So $h/y^k \in \mathbb{K}[x, x^{-1}]$ and then $h(x, y) = x^{-r}y^k P(x)$ for some $r \in \mathbb{N}$ and $P(z) \in \mathbb{K}[z]$. Therefore,

$$p(x, y) = h(x^\lambda y^\mu, x^t y^{-s}) = (x^\lambda y^\mu)^{-r}(x^t y^{-s})^k P(x^\lambda y^\mu) = x^{-\lambda r + tk} y^{-\mu r - sk} P(x^\lambda y^\mu).$$

Letting $\alpha = -\lambda r + tk$ and $\beta = -\mu r - sk$ gives $p(x, y) = x^\alpha y^\beta P(x^\lambda y^\mu)$. □

Combining [1, Theorem 10] and [26, Theorem 4.6], we have the following existence criteria for telescopers.

Theorem 5.5. *Let T be a (σ_x, σ_y)-hypergeometric term whose σ_y-quotient has a σ_y-standard kernel K and a corresponding shell S, and let r be a σ_y-remainder of S with respect to K. Then T has a telescoper if and only if the significant denominator of r is integer-linear.*

When telescopers exist, we are then able to use the reduction algorithm developed in Section 3 to construct a telescoper for a given (σ_x, σ_y)-hypergeometric term effectively.

HypergeomTelescoping. Given a (σ_x, σ_y)-hypergeometric term T, compute a telescoper of minimal order for T and its certificate if T has telescopers.

1. Apply **HypergeomReduction** to T with respect to y to find a (σ_x, σ_y)-hypergeometric term H whose σ_y-quotient K is σ_y-standard, and two rational functions $g_0, r_0 \in \mathbb{F}(y)$ such that
$$T = \Delta_y(g_0 H) + r_0 H, \tag{31}$$
and r_0 is a σ_y-remainder with respect to K. If $r_0 = 0$ then return $(1, g_0 H)$.

2. If the significant denominator of r_0 is not integer-linear, then return "No telescoper exists!".

3. Set $N = \sigma_x(H)/H$ and $r = \ell_0 r_0$, where ℓ_0 is an indeterminate.
For $i = 1, 2, \ldots$ do

3.1 Apply **HypergeomReduction** to $\sigma_x(r_{i-1})NH$ with respect to y, with the choice of the σ_y-standard kernel K and the shell $\sigma_x(r_{i-1})N$ in its step 1, to find $\tilde{g}_i \in \mathbb{F}(y)$ and a σ_y-remainder $\tilde{r}_i$ with respect to K such that
$$\sigma_x(r_{i-1})NH = \Delta_y(\tilde{g}_i H) + \tilde{r}_i H. \tag{32}$$

3.2 Apply **RemainderLinearization** to $\tilde{r}_i$ with respect to r and K to find $\bar{g}_i \in \mathbb{F}(y)$ and another σ_y-remainder r_i with respect to K such that
$$\tilde{r}_i = \Delta_K(\bar{g}_i) + r_i \tag{33}$$
and $r + \ell_i r_i$ is a σ_y-remainder with respect to K, where ℓ_i is an indeterminate.

3.3 Set $g_i = \sigma_x(g_{i-1})N + \tilde{g}_i + \bar{g}_i$ and update r to be $r + \ell_i r_i$.

3.4 Find $\ell_0, \ldots, \ell_i \in \mathbb{F}$ such that $r = 0$ by solving a linear system in $\ell_0, \ldots, \ell_i$ over $\mathbb{F}$. If there is a nontrivial solution, return $(\sum_{j=0}^{i} \ell_j S_x^j,\ \sum_{j=0}^{i} \ell_j g_j H)$.

Remark 5.6. *The algorithm* **HypergeomTelescoping** *separates the computation of telescopers from that of certificates. In applications where certificates are irrelevant, we can drop all computations related to the preimages of Δ_y. In particular, all rational functions g_i can be discarded and we do not need to calculate $\sum_{j=0}^{i} \ell_j g_j H$ in the end.*

Theorem 5.7. *For a (σ_x, σ_y)-hypergeometric term T, the algorithm* **HypergeomTelescoping** *terminates and correctly finds a telescoper of minimal order for T and a corresponding certificate when such telescopers exist.*

Proof. By Theorem 3.18, $r_0 = 0$ implies that T is σ_y-summable and thus 1 is a telescoper of minimal order for T. Together with Theorem 5.5, we see that steps 1 and 2 are correct.

Now assume that the algorithm proceeds to step 3. Then T has a telescoper of order at least one. It follows from (31) and $\sigma_x(H) = NH$ that

$$\sigma_x(T) = \Delta_y(\sigma_x(g_0)NH) + \sigma_x(r_0)NH.$$

By viewing $\sigma_x(r_0)NH$ as a (σ_x, σ_y)-hypergeometric term whose σ_y-quotient has a σ_y-standard kernel K and a corresponding shell $\sigma_x(r_0)N$, we can perform **HypergeomReduction** to $\sigma_x(r_0)NH$ to obtain $\tilde{g}_1 \in \mathbb{F}(y)$ and a σ_y-remainder $\tilde{r}_1$ with respect to K so that (32) holds for $i = 1$. According to Theorem 4.1, the algorithm **RemainderLinearization** enables us to find $\bar{g}_1 \in \mathbb{F}(y)$ and another σ_y-remainder r_1 with respect to K such that (33) holds for $i = 1$ and $r + \ell_1 r_1 = \ell_0 r_0 + \ell_1 r_1$ for all $\ell_0, \ell_1 \in \mathbb{F}$ is again a σ_y-remainder with respect to K. Setting $g_1 = \sigma_x(g_0)N + \tilde{g}_1 + \bar{g}_1$, we thus get $\sigma_x(T) = \Delta_y(g_1 H) + r_1 H$. By a direct induction on $i \in \mathbb{Z}^+$, we see that

$$\sigma_x^i(T) = \Delta_y(g_i H) + r_i H \tag{34}$$

holds in the loop of step 3 every time the algorithm passes through step 3.3. Moreover, $r = \sum_{j=0}^{i} \ell_j r_j$ for all $\ell_j \in \mathbb{F}$ is a σ_y-remainder with respect to K.

Let $\rho \in \mathbb{Z}^+$ and define $L = \sum_{i=0}^{\rho} c_i S_x^i$ with $c_i \in \mathbb{F}$ and $c_\rho \neq 0$. Then by (34),

$$L(T) = \Delta_y\Big(\sum_{i=0}^{\rho} c_i g_i H\Big) + \Big(\sum_{i=0}^{\rho} c_i r_i\Big)H.$$

Since $\sum_{i=0}^{\rho} c_i r_i$ is a σ_y-remainder with respect to K, we conclude from Theorem 3.18 that L is a telescoper for T if and only if $\sum_{i=0}^{\rho} c_i r_i$ is equal to zero, which happens if and only if the linear homogeneous system in $\ell_0, \ldots, \ell_\rho$ over $\mathbb{F}$ obtained by equating $\sum_{i=0}^{\rho} \ell_i r_i$ to zero has a nontrivial solution $\ell_0 = c_0, \ldots, \ell_\rho = c_\rho$. Therefore, the first linear dependency among the r_i gives rise to a telescoper of minimal order. □

Example 5.8. *Assume that σ_x and σ_y are both the usual shift operators. Let $T(x, y) = \binom{x+2y}{y}$. Then T is a (σ_x, σ_y)-hypergeometric term whose σ_y-quotient has a σ_y-standard kernel $K = (x + 2y + 1)(x + 2y + 2)/((y + 1)(x + y + 1))$ and a corresponding shell $S = 1$. So $H = T/S = T$. Applying the algorithm* **HypergeomTelescoping** *to T, we obtain in step 3 that*

$$\sigma_x^i(T) = \Delta_y(g_i H) + \frac{p_i}{v} H \quad \textit{for } i = 0, 1, 2,$$

where $v = (y + 1)(x + y + 1)$, $p_0 = (-x^2 + x + 2y + 2)/3$, $p_1 = (-2x^2 - 3xy - x + y + 1)/3$, $p_2 = (-x^2 - 3xy - 2x - y - 1)/3$, and $g_i \in \mathbb{F}(y)$ are not displayed here to keep things neat. By finding an $\mathbb{F}$-linear dependency among p_0, p_1, p_2, we get

$$L = S_x^2 - S_x + 1$$

is a telescoper of minimal order for T.

Example 5.9. *Assume that σ_x and σ_y are both the q-shift operators. Let $T(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$. Then T is a (σ_x, σ_y)-hypergeometric term with $\sigma_x(T(n,k)) = T(n+1,k)$, $\sigma_y(T(n,k)) = T(n,k+1)$ and $q^n = x$, $q^k = y$. The σ_y-quotient of T has a σ_y-standard kernel $K = (x-y)/(y(qy-1))$ and a corresponding shell $S = 1$. So $H = T/S = T$. Applying the algorithm* **HypergeomTelescoping** *to T, we obtain in step 3 that*

$$\sigma_x^i(T) = \Delta_y(g_i H) + \frac{p_i}{v} H \quad \text{for } i = 0, 1, 2,$$

where $v = y(qy-1)$, $p_0 = x - y$, $p_1 = (qx-1)y$, $p_2 = (qx-1)(x+y)$, and $g_i \in \mathbb{F}(y)$ are not displayed here to keep things neat. By finding an $\mathbb{F}$-linear dependency among p_0, p_1, p_2, we get

$$L = S_x^2 - 2S_x + 1 - q^{n+1}$$

is a telescoper of minimal order for T.

6 Implementation and applications

We have implemented our algorithms in the computer algebra system Maple 2020. The code and its demo are available via the link:

http://www.mmrc.iss.ac.cn/~schen/CDGHL2025.html

We aim to compare their runtime and memory requirements with the performance of known algorithms so as to get an idea about the efficiency. Since such experiments for the shift case have been well conducted in [23], we will focus on the q-shift case in this section. All timings are measured in seconds on a macOS computer with 32GB RAM and 2.3 GHz Quad-Core Intel Core i7 processors. The computations for the experiments did not use any parallelism.

We consider bivariate q-hypergeometric terms of the form

$$T = \frac{f(q^n, q^k)}{g(q^{n+k})} \frac{(q;q)_{2\alpha n+k}}{(q;q)_{n+\alpha k}},$$

where $f \in \mathbb{Q}(q)[q^n, q^k]$ is of total degree 1, $g = p\sigma_z^\lambda(p)\sigma_z^\mu(p)$ with $p \in \mathbb{Q}[z]$ of degree d and $\alpha, \lambda, \mu \in \mathbb{N}$. For a selection of random terms of this type for different choices of $(d, \alpha, \lambda, \mu)$, Table 1 compares the timings of Maple's implementation of q-Zeilberger's algorithm (Z) and two variants of the algorithm **HypergeomTelescoping** in the q-shift case from Section 5: For the column HTC we computed both the telescoper and the certificate, and for HT only the telescoper was returned. The difference between these two variants mainly comes from the time needed to bring the rational part r in the certificate rH on a common denominator. When it is acceptable to keep the rational part as an unnormalized linear combination of rational functions, the time is virtually the same as that for HT.

Our implementation enhances the applicability of (q-)Zeilberger's algorithm, as illustrated by the following example.

Example 6.1. *Stanton [44] conjectured the following identity*

$$\sum_k (-1)^k q^{4k^2} \begin{bmatrix} 2n \\ n-4k \end{bmatrix}_q = \sum_k q^{2k^2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q (-q;q^2)_{n-2k}(-1;q^4)_k, \tag{35}$$

which was proved by Paule and Riese [40] using q-Zeilberger's algorithm. They observed that taking the summation range to be {k, −Infinity, Infinity} *saves computing time. Actually, due to the natural boundary of Gaussian binomial coefficients, we can further omit the computation of certificates and write down recurrence equations for both sides of (35) merely using telescopers.*

$(d, \alpha, \lambda, \mu)$	Z	HTC	HT	order
(1, 1, 1, 5)	1.05	.94	.50	2
(1, 2, 1, 5)	5082.63	1854.00	797.41	5
(2, 1, 1, 5)	17.36	7.37	3.15	3
(2, 2, 1, 5)	167299.50	18774.25	9778.94	6
(3, 1, 1, 5)	884.95	72.20	22.98	4
(1, 1, 5, 10)	17.40	8.44	2.53	2
(1, 2, 5, 10)	39997.13	18542.85	9139.08	5
(1, 1, 10, 15)	60.93	48.97	7.92	2

Table 1: Comparison of q-Zeilberger's algorithm to reduction-based creative telescoping with and without construction of a certificate for a collection of random terms

For the summand in either side of (35)*, without computing the certificate, our implementation returns the same telescoper*

$$S_x^3-(q^{2n+5}+q^{2n+4}+q^{2n+3}+1)S_x^2+q^{2n+4}(q^{2n+3}+q^{2n+2}+q^{2n+1}-1)S_x-q^{2n+3}(q^{2n+2}-1)(q^{2n+1}-1),$$

where $q^n = x$*, thus implying that both sides of* (35) *satisfy the same recurrence equation*

$$f(n+3) - (q^{2n+5} + q^{2n+4} + q^{2n+3} + 1)f(n+2) + q^{2n+4}(q^{2n+3} + q^{2n+2} + q^{2n+1} - 1)f(n+1) \\ -q^{2n+3}(q^{2n+2} - 1)(q^{2n+1} - 1)f(n) = 0.$$

Checking the identity at the initial values $n = 0, 1, 2$ *completes the proof of* (35)*.*

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References

[1] Sergei A. Abramov. When does Zeilberger's algorithm succeed? *Adv. in Appl. Math.*, 30(3):424–441, 2003.

[2] Sergei A. Abramov, Peter Paule, and Marko Petkovšek. q-Hypergeometric solutions of q-difference equations. *Discrete Math.*, 180(1-3):3–22, 1998.

[3] Sergei A. Abramov and Marko Petkovšek. Minimal decomposition of indefinite hypergeometric sums. In *Proceedings of ISSAC'01*, pages 7–14. ACM, New York, 2001.

[4] Sergei A. Abramov and Marko Petkovšek. On the structure of multivariate hypergeometric terms. *Adv. in Appl. Math.*, 29(3):386–411, 2002.

[5] Sergei A. Abramov and Marko Petkovšek. Rational normal forms and minimal decompositions of hypergeometric terms. *J. Symbolic Comput.*, 33(5):521–543, 2002.

[6] George E. Andrews. *The Theory of Partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[7] George E. Andrews. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, volume 66 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[8] George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[9] Moa Apagodu and Doron Zeilberger. Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory. *Adv. in Appl. Math.*, 37(2):139–152, 2006.

[10] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li. Complexity of creative telescoping for bivariate rational functions. In *Proceedings of ISSAC'10*, pages 203–210. ACM, New York, 2010.

[11] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, Ziming Li, and Guoce Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *Proceedings of ISSAC'13*, pages 77–84. ACM, New York, 2013.

[12] Alin Bostan, Frédéric Chyzak, Pierre Lairez, and Bruno Salvy. Generalized Hermite reduction, creative telescoping and definite integration of D-finite functions. In *Proceedings of ISSAC'18*, pages 95–102. ACM, New York, 2018.

[13] Alin Bostan, Pierre Lairez, and Bruno Salvy. Creative telescoping for rational functions using the Griffiths-Dwork method. In *Proceedings of ISSAC'13*, pages 93–100. ACM, New York, 2013.

[14] Hadrien Brochet and Bruno Salvy. Reduction-based creative telescoping for definite summation of D-finite functions. *J. Symbolic Comput.*, 125:102329, 2024.

[15] Manuel Bronstein. *Symbolic Integration I: Transcendental Functions*, volume 1 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2005.

[16] Manuel Bronstein, Ziming Li, and Min Wu. Picard-Vessiot extensions for linear functional systems. In *Proceedings of ISSAC'05*, pages 68–75. ACM, New York, 2005.

[17] Shaoshi Chen. A reduction approach to creative telescoping. In *Proceedings of ISSAC'19*, pages 11–14. ACM, New York, 2019.

[18] Shaoshi Chen, Lixin Du, and Manuel Kauers. Lazy Hermite reduction and creative telescoping for algebraic functions. In *Proceedings of ISSAC'21*, pages 75–82. ACM, New York, 2021.

[19] Shaoshi Chen, Lixin Du, and Manuel Kauers. Hermite reduction for D-finite functions via integral bases. In *Proceedings of ISSAC'23*, pages 155–163. ACM, New York, 2023.

[20] Shaoshi Chen, Lixin Du, Manuel Kauers, and Rong-Hua Wang. Reduction-based creative telescoping for P-recursive sequences via integral bases. *J. Symbolic Comput.*, 126:102341, 2025.

[21] Shaoshi Chen, Mark van Hoeij, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for fuchsian D-finite functions. *J. Symbolic Comput.*, 85:108–127, 2018.

[22] Shaoshi Chen, Qing-Hu Hou, Hui Huang, George Labahn, and Rong-Hua Wang. Constructing minimal telescopers for rational functions in three discrete variables. *Adv. in Appl. Math.*, 141:102389, 2022.

[23] Shaoshi Chen, Hui Huang, Manuel Kauers, and Ziming Li. A modified Abramov-Petkovšek reduction and creative telescoping for hypergeometric terms. In *Proceedings of ISSAC'15*, pages 117–124. ACM, New York, 2015.

[24] Shaoshi Chen, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for algebraic functions. In *Proceedings of ISSAC'16*, pages 175–182. ACM, New York, 2016.

[25] Shaoshi Chen and Michael F. Singer. On the summability of bivariate rational functions. *J. Algebra*, 409:320–343, 2014.

[26] William Y. C. Chen, Qing-Hu Hou, and Yan-Ping Mu. Applicability of the q-analogue of Zeilberger's algorithm. *J. Symbolic Comput.*, 39(2):155–170, 2005.

[27] Hao Du, Hui Huang, and Ziming Li. A q-analogue of the modified Abramov-Petkovšek reduction. In *Advances in Computer Algebra*, volume 226 of *Springer Proc. Math. Stat.*, pages 105–129. Springer, Cham, 2018.

[28] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.

[29] Mark Giesbrecht, Hui Huang, George Labahn, and Eugene Zima. Efficient integer-linear decomposition of multivariate polynomials. In *Proceedings of ISSAC'19*, pages 171–178. ACM, New York, 2019.

[30] Mark Giesbrecht, Hui Huang, George Labahn, and Eugene Zima. Efficient q-integer linear decomposition of multivariate polynomials. *J. Symbolic Comput.*, 107:122–144, 2021.

[31] Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.

[32] Joris van der Hoeven. Creative telescoping using reductions, 2018. Preprint: hal-01773137.

[33] Joris van der Hoeven. Constructing reductions for creative telescoping: the general differentially finite case. *Appl. Algebra Engrg. Comm. Comput.*, 32(5):575–602, 2021.

[34] Qing-Hu Hou. k-free recurrences of double hypergeometric terms. *Adv. in Appl. Math.*, 32(3):468–484, 2004.

[35] Hui Huang. New bounds for hypergeometric creative telescoping. In *Proceedings of ISSAC'16*, pages 279–286. ACM, New York, 2016.

[36] Michael Karr. Summation in finite terms. *J. Assoc. Comput. Mach.*, 28(2):305–350, 1981.

[37] Tom H. Koornwinder. On Zeilberger's algorithm and its q-analogue. *J. Comput. Appl. Math.*, 48(1-2):91–111, 1993.

[38] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[39] Mohamud Mohammed and Doron Zeilberger. Sharp upper bounds for the orders of the recurrences output by the Zeilberger and q-Zeilberger algorithms. *J. Symbolic Comput.*, 39(2):201–207, 2005.

[40] Peter Paule and Axel Riese. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping. In *Special functions, q-series and related topics (Toronto, ON, 1995)*, volume 14 of *Fields Inst. Commun.*, pages 179–210. Amer. Math. Soc., Providence, RI, 1997.

[41] Peter Paule and Volker Strehl. Symbolic summation - some recent developments. In *Computer Algebra in Science and Engineering - Algorithms, Systems, and Applications*, J. Fleischer, J. Grabmeier, F. Hehl, and W. Kuechlin, eds., pages 138–162. World Scientific, Singapore, 1995.

[42] Marko Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.*, 14(2-3):243–264, 1992.

[43] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger. $A = B$. A K Peters, Ltd., Wellesley, MA, 1996.

[44] Dennis Stanton. *Talk* at the Workshop "Special Functions, q-Series and Related Topics", organized by the Fields Institute for Research in Mathematical Sciences at University College, 12–23 June 1995, Toronto, Ontario.

[45] Steven H. Weintraub. *Galois Theory.* Universitext. Springer, New York, second edition, 2009.

[46] Doron Zeilberger. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.*, 80(2):207–211, 1990.

[47] Doron Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32(3):321–368, 1990.

[48] Doron Zeilberger. The method of creative telescoping. *J. Symbolic Comput.*, 11(3):195–204, 1991.